A Generalized Hamiltonian Dynamics
for Relativistic Particles with Spin - I.

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1. – Introduction.

In classical mechanics a particle is considered to be a point particle, that is, without extension in space. For a free particle it is a well-known result that the angular momentum L is conserved.

In Dirac's theory of the electron (in which the electron is considered to be a point particle) the corresponding quantity is not conserved. To preserve the conservation theorem a generalized angular momentum M is ascribed to the particle with $M = L + S$, L being the quantity corresponding to the classical angular momentum and S a quantity chosen so that M is conserved. L is called the orbital angular momentum and S the spin angular momentum of the particle.

By considering, instead of the motion of a particle, the motion of a fluid

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droplet in terms of a conserved energy-momentum tensor which is not symmetric WEYSSENHOFF [1] showed that, for the droplet, it is the sum of external and internal angular momentum densities that is conserved.

The former is called the moment of the linear momentum density, that is, it corresponds to the angular momentum of the point particle. The latter is called the density of spin and its existence is connected with the asymmetry of the energy-tensor.

Weyssenhoff's theory has been extended by BOHM and VIGIER [2]. They have given a physical interpretation to the theory and have shown that Weyssenhoff's postulate, that the time components of the spin density tensor vanish in the rest system, is not necessary but defines a particular type of motion. BOHM and VIGIER assume that certain average motions of the particle can be described by a symmetric energy momentum tensor density which vanishes outside a space-like three dimensional limited region. Two points are defined within the particle—the centre of mass and the centre of matter density—and the spin angular momentum is shown to be the total angular momentum of the finite particle with respect to the centre of matter density.

To formulate a generalized Hamiltonian dynamics for particles with spin we observe that for the free relativistic point particle, if the world momentum p_α is not equal to a constant (mass) times the world velocity $\mathrm{d}x_\alpha/\mathrm{d}\tau$ (where τ is the proper time) angular momentum will not be conserved.

For particles with spin, defining the momentum by

$$(1) \qquad p_\alpha = \frac{\mathrm{d}x_\alpha}{\mathrm{d}\tau} + g_\alpha$$

leads to non-conservation of angular momentum for the free particle and the possibility of introducing spin. If relation (1) holds however the motion of the particles cannot be described in terms of position co-ordinates and momenta alone. Thus to introduce spin we must introduce extra dynamical variables for the particles. This will be done by considering particles as finite and not as point particles.

It is well known in classical mechanics that the general motion of a fluid element can be considered as the resultant of a rigid body motion together with a deformation of the element. A rigid body motion can be considered as the translation of any specified point P of the body together with a rotation about some axis through P.

By analogy we shall consider some point P within the particle and discuss the motion in terms of the motion of P and the motion relative to P.

To do this we will consider a frame of reference whose origin is P and which

[1] J. WEYSSENHOFF and A. RAABE: *Acta Phys. Polon.*, **9**, 7 (1947).
[2] D. BOHM and J. P. VIGIER: *Phys. Rev.*, **109**, 1882 (1952).

moves with P ([3]). To start with at least P will not be specified. Let U^μ_α ($\mu = 0, 1, 2, 3$; $\alpha = 1, 2, 3, 4$) be the components, relative to a Lorentz frame $\mathscr{L}$, of a tetrad of unit vectors U^μ which are orthogonal and with U^0 a vector tangential to the world line of P in $\mathscr{L}$. The U^μ_α will be considered as functions of the length ξ of the world line of P. These will be considered as the dynamical variables in a generalized form of Hamiltonian dynamics ([3]).

2. – Generalized hamiltonian dynamics.

Generalized Hamiltonian equations for a classical system have been obtained by MARTIN ([4]). MARTIN discussed a system defined by N dynamical variables $\varphi_1, \varphi_2, ..., \varphi_N$.

The Poisson bracket of two functions u and v of the φ_N is defined by ([5])

$$(2.1) \qquad (u, v) = \frac{\partial u}{\partial \varphi_R} \alpha_{RS} \frac{\partial v}{\partial \varphi_S} ,$$

where the N^2 functions $\alpha_{RS}(\varphi_T)$ satisfy

$$(2.2) \qquad \begin{cases} \alpha_{RS} + \alpha_{SR} = 0 , \\ \alpha_{AS} \dfrac{\partial \alpha_{BC}}{\partial \varphi_S} + \alpha_{BS} \dfrac{\partial \alpha_{CA}}{\partial \varphi_S} + \alpha_{CS} \dfrac{\partial \alpha_{AB}}{\partial \varphi_S} = 0 . \end{cases}$$

These are necessary and sufficient conditions for the P.b. to satisfy all the usual conditions including the Jacobi identity.

The time derivative of any function $u(\varphi, t)$ will be given by

$$(2.3) \qquad \frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\partial u}{\partial t} + (u, H) ,$$

where H, the Hamiltonian, is chosen to represent the development of the system in time.

([3]) A four dimensional set of axes moving with the particle has been used to give a Lagrangian formulation of Bohm and Vigier's theory see F. HALBWACHS, P. HILLION and J. P. VIGIER: *Nuovo Cimento*, **10**, 817 (1958); F. HALBWACHS and J. P. VIGIER: *Comp. Rend.*, **248**, 1124 (1959). Tetrads associated with the particle have also been considered by T. TAKABAYASI: *Nuovo Cimento*, **13**, 532 (1959).

([4]) J. L. MARTIN: *Proc. Roy. Soc.*, A **251**, 536 (1959).

([5]) The summation convention for repeated suffixes is used.

In particular

(2.4)
$$\frac{\mathrm{d}}{\mathrm{d}t}\,\varphi_R = (\varphi_R,\,H) = \alpha_{RS}\,\frac{\partial H}{\varrho\varphi_S}\,.$$

Eq. (2.4) are the generalized Hamiltonian equations.

3. – The equations of motion for the $U^\mu{}_\alpha$

The $U^\mu{}_\alpha$ are orthogonal unit vectors. Therefore

(3.1)
$$U^\mu{}_\alpha U^\nu{}_\alpha = \delta^{\mu\nu}\,.$$

Consider U a 4×4 matrix where $U_{\mu\alpha} = U^\mu{}_\alpha$. Then eq. (3.1) can be written as the matrix equation

$$U\widetilde{U} = 1\,,$$

where $\sim$ denotes the transposed matrix.

Obviously $\widetilde{U}U = 1$ which gives the equations

(3.2)
$$U^\mu{}_\alpha U^\mu{}_\beta = \delta_{\alpha\beta}\,.$$

Consider now the tetrads $U'^\mu{}_\alpha$, $U^\mu{}_\alpha$ at two successive positions on the world line of P. These will be related by an infinitesimal transformation

$$U'^\mu{}_\alpha = U^\mu{}_\alpha + \delta\omega_{\mu\nu}U^\nu{}_\alpha$$

(with $\delta\omega_{\mu\nu} + \delta\omega_{\nu\mu} = 0$) or

$$\delta U^\mu{}_\alpha = \delta\omega_{\mu\nu}U^\nu{}_\alpha\,.$$

Dividing by $\delta\xi$ and taking the limit as $\delta\xi \to 0$ we get

(3.3)
$$U^\mu{}_\alpha = \frac{\mathrm{d}U^\mu{}_\alpha}{\mathrm{d}\xi} = \Omega_{\mu\nu}\,U^\nu{}_\alpha\,,$$

where

$$\Omega_{\mu\nu} = \frac{\mathrm{d}\omega_{\mu\nu}}{\mathrm{d}\xi}\,,$$

so that

(3.4)
$$\Omega_{\mu\nu} + \Omega_{\nu\mu} = 0\,.$$

Eq. (3.3) will be regarded as the equations of motion for the $U^\mu{}_\alpha$.

If x_α be the co-ordinates of P in $\mathscr{L}$ then

$$\mathrm{d}\xi^2 = \mathrm{d}x_\alpha\,\mathrm{d}x_\alpha$$

or

$$\frac{\mathrm{d}x_\alpha}{\mathrm{d}\xi}\frac{\mathrm{d}x_\alpha}{\mathrm{d}\xi} = 1\,.$$

Thus

$$U^0_\alpha = \frac{\mathrm{d}x_\alpha}{\mathrm{d}\xi}\,.$$

In our generalized Hamiltonian dynamics the variables will be the x_α and the U^μ_α—twenty variables in all of which only ten are independent on account of eq. (3.1)—with the arc length ξ (of the world line of P) the independent parameter. .

4. – The generalized hamiltonian dynamics for the finite particle.

In non relativistic mechanics Hamilton's equations are

$$\frac{\mathrm{d}x_r}{\mathrm{d}t} = \frac{\partial H}{\partial p_r}, \qquad \frac{\mathrm{d}p_r}{\mathrm{d}t} = -\frac{\partial H}{\partial x_r}, \qquad\qquad (r = 1, 2, 3)\,,$$

and for the free point particle

$$H = \frac{1}{2m}\,p_r p_r\,.$$

The P.b. of two functions u and v of the x_r, p_r is defined by

$$(u,\,v) = \frac{\partial u}{\partial x_r}\frac{\partial v}{\partial p_r} - \frac{\partial u}{\partial p_r}\frac{\partial v}{\partial x_r}\,.$$

This scheme can be obtained from Martin's equations (with $N = 6$) by writing

$$\varphi_R = \varphi_{3A+r} \qquad\qquad (R = 1, 2, ..., 6;\; A = 0, 1)\,,$$

with

$$\varphi_R \equiv x_r \qquad\qquad\qquad \text{for } A = 0\,,$$

$$\varphi_R \equiv p_r \qquad\qquad\qquad \text{for } A = 1\,,$$

and choosing

$$\alpha_{RS} = \varepsilon_{A\Phi}\, \delta_{rs} \qquad\qquad (S = 3\Phi + s; \;\; \Phi = 0, 1) \,,$$

where

$$\varepsilon_{01} = -\,\varepsilon_{10} = 1 \,, \qquad \varepsilon_{00} = \varepsilon_{11} = 0 \,.$$

Similarly for the relativistic point particle, for which

$$\frac{\mathrm{d}x_\alpha}{\mathrm{d}\tau} = \frac{\partial H}{\partial p_\alpha}\,, \qquad\qquad \frac{\mathrm{d}p_\alpha}{\mathrm{d}\tau} = -\,\frac{\partial H}{\partial x_\alpha} \qquad\qquad (\alpha = 1, 2, 3, 4)$$

(where τ is the proper time),

(4.1) $$H = \frac{1}{2m}\, p_\alpha p_\alpha \;(^6) \,,$$

$$(u,\, v) = \frac{\partial u}{\partial x_\alpha}\, \frac{\partial v}{\partial p_\alpha} - \frac{\partial u}{\partial p_\alpha}\, \frac{\partial v}{\partial x_\alpha}\,,$$

write

$$\varphi_R = \varphi_{4\,A+\alpha} \qquad\qquad (R = 1, 2, ..., 8; \; A = 0, 1)$$

and choose

$$\alpha_{RS} = \varepsilon_{A\Phi}\, \delta_{\alpha\beta} \,.$$

This suggests that, for the twenty dynamical variable case, we write

(4.2a) $$\varphi_R = \varphi_{4\,A+\alpha} \qquad (A = 0, 1, 2, 3, 4; \; \alpha = 1, 2, 3, 4) \,,$$

with

(4.2b) $$\begin{cases} \varphi_R \equiv x_\alpha & \text{for } A = 4 \,, \\ \varphi_R \equiv U^\mu{}_\alpha & \text{for } A = \mu \,, \end{cases}$$

and choose

(4.3) $$\alpha_{RS} = \varepsilon_{A\Phi}\, \delta_{\alpha\beta} \,.$$

The generalized Hamiltonian for the free particle will be defined by

(4.4) $$H = \frac{m}{2} \sum \varphi_R \varphi_R \,,$$

$(^6)$ More generally, if $\lambda^2 = (1/m^2) p_\alpha p_\alpha (= c^2)$ we could have

$$H = mcF(\lambda) \,,$$

with $F(\lambda)$ an arbitrary function satisfying $F'(\lambda)_{\lambda=c} = 1.$

where m is a constant and the summation is over all the φ_R except $\varphi_R = x_\alpha$. Thus

$$(4.5) \qquad\qquad H = \tfrac{1}{2} m\, U^\mu{}_\alpha U^\mu{}_\alpha$$

which is an obvious generalization of (4.1).

With α_{RS} given by (4.2) eq. (2.1) becomes

$$(4.6) \qquad (u,\, v) = \varepsilon_{4\mu} \left[\frac{\partial u}{\partial x_\alpha} \frac{\partial v}{\partial U^\mu{}_\alpha} - \frac{\partial u}{\partial U^\mu{}_\alpha} \frac{\partial v}{\partial x_\alpha} \right] + \varepsilon_{\mu\nu} \frac{\partial u}{\partial U^\mu{}_\alpha} \frac{\partial u}{\partial U^\nu{}_\alpha},$$

$$(\mu,\, \nu = 0, 1, 2, 3; \ \alpha = 1, 2, 3, 4),$$

and (4.2b) in (2.4) gives

$$(4.7) \qquad\qquad \frac{\mathrm{d}x_\alpha}{\mathrm{d}\xi} = (x_\alpha,\, H) = m\varepsilon_{4\mu} U^\mu{}_\alpha,$$

$$(4.8) \qquad\qquad \frac{\mathrm{d}U^\mu{}_\alpha}{\mathrm{d}\xi} = (U^\mu{}_\alpha,\, H) = m\varepsilon_{\mu\nu} U^\nu{}_\alpha.$$

Comparing (4.7) and (4.8) with (3.3) and (3.5) we see that

$$(4.9) \qquad \begin{cases} m\,\varepsilon_{4\lambda} U^\lambda{}_\alpha = U^0{}_\alpha \\ \text{and} \\ m\,\varepsilon_{\mu\nu} U^\nu{}_\alpha = \Omega_{\mu\nu} U^\nu{}_\alpha. \end{cases}$$

Thus the P.b. is completely defined

$$(4.10) \qquad m(u,\, v) = \frac{\partial u}{\partial x_\alpha} \frac{\partial v}{\partial U^0{}_\alpha} - \frac{\partial u}{\partial U^0{}_\alpha} \frac{\partial v}{\partial x_\alpha} + \Omega_{\mu\nu} \frac{\partial u}{\partial U^\mu{}_\alpha} \frac{\partial v}{\partial U^\nu{}_\alpha}.$$

This, together with

$$\frac{\mathrm{d}u}{\mathrm{d}\xi} = \frac{\partial u}{\partial \xi} + (u,\, H),$$

where

$$H = \tfrac{1}{2} m\, U^\mu{}_\alpha U^\mu{}_\alpha$$

specifies the dynamics scheme for the free finite particle.

Note that for the point particle $m U^0{}_\alpha = p_\alpha$ so that the generalized P.b. (4.10) is obtained from that for the point particle by adding the term $\Omega_{\mu\nu} (\partial u / \partial u^\mu{}_\alpha)(\partial v / \partial u^\nu{}_\alpha)$ which will be non-zero only if u and v are functions of the « new » variables $U^1{}_\alpha$, $U^2{}_\alpha$, $U^3{}_\alpha$. Thus point dynamics is the limiting case of our dynamics.

5. – The momentum and spin of the finite particle.

Having introduced extra dynamical variables and the dynamics scheme they satisfy, we can now define a generalized momentum for the finite particle of the type given by eq. (1.1) and therefore introduce spin angular momentum.

It will be assumed that the motion of the finite particle can be described in terms of its momentum p_α, its external angular momentum $L_{\alpha\beta}$, and its spin, angular momentum $S_{\alpha\beta}$. These will be defined by the equations

$$(5.1) \qquad\qquad p_\alpha = a_\mu U^\mu{}_\alpha$$

(a_μ = multipliers; we assume that a_0 will have the dimensions of mass),

$$(5.2) \qquad\qquad L_{\alpha\beta} = x_\alpha p_\beta - x_\beta p_\alpha \,,$$

$$(5.3) \qquad\qquad \dot S_{\alpha\beta} = s_{\mu\nu} U^\mu{}_\alpha U^\nu{}_\beta \qquad\qquad (s_{\mu\nu} = \text{multipliers})$$

with

$$s_{\mu\nu} + s_{\nu\mu} = 0 \,.$$

For the free particle these satisfy the conservation laws

$$(5.4) \qquad\qquad \frac{\mathrm{d}p_\alpha}{\mathrm{d}\xi} = (p_\alpha, H) = 0 \,,$$

$$(5.5) \qquad\qquad \frac{\mathrm{d}M_{\alpha\beta}}{\mathrm{d}\xi} = \frac{\mathrm{d}}{\mathrm{d}\xi}(L_{\alpha\beta} + S_{\alpha\beta}) = 0 \,,$$

$M_{\alpha\beta} = L_{\alpha\beta} + S_{\alpha\beta}$ is called the total angular momentum [7].

Moreover, if we assume that the same P.b. relations hold between the x_α, p_α, $M_{\alpha\beta}$, as hold between the corresponding quantities for the classical point particle; and that the a_μ and the $s_{\mu\nu}$ for the free particle are independent of position along the world line of P then these latter will be determined completely in terms of the $\Omega_{\mu\nu}$. This means that the motion of the particle can be described in terms of the motion of the point P and the rotation of a four-dimensional set of axes whose origin is P. In particular it will

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[7] Similarly p_α could be considered as the total linear momentum, being the sum of the external linear momentums $a_0 U^0{}_\alpha$ and the internal momentum $a_i U^i{}_\alpha$.

be shown that for this set of axes we could take the set of vectors whose direction cosines satisfy the Frenet formulae in four dimensions. This set we shall call the Frenet frame.

6. – Determination of the a_μ, $s_{\mu\nu}$.

We have the following equations to determine the a_μ, $s_{\mu\nu}$

(6.1) $$(p_\alpha, H) = 0$$

(6.2) $$(M_{\alpha\beta}, H) = 0$$

(6.3a) $$(x_\alpha, p_\beta) = \delta_{\alpha\beta}$$

(6.3b) $$(x_\alpha, x_\beta) = (p_\alpha, p_\beta) = 0$$

(6.4) $$(M_{\alpha\beta}, M_{\mu\nu}) = M_{\alpha\mu}\delta_{\beta\nu} + M_{\beta\nu}\delta_{\alpha\mu} - M_{\alpha\nu}\delta_{\beta\mu} - M_{\beta\mu}\delta_{\alpha\nu} .$$

i) $(p_\alpha, H) = 0$.

$$\Omega_{\mu\nu} \frac{\partial p_\alpha}{\partial U^\mu_\beta} \frac{\partial H}{\partial U^\nu_\beta} = 0 ,$$

that is

$$\Omega_{\mu\nu} a_\mu U^\nu_\alpha = 0 .$$

Multiplying by U^σ_α and summing over α we get

(6.5) $$\Omega_{\mu\sigma} a_\mu = 0 .$$

ii) $(M_{\alpha\beta}, H) = 0$.

$$(x_\alpha p_\beta - x_\beta p_\alpha + s_{\mu\nu} U^\mu_\beta U^\nu_\alpha, H) = 0 ,$$

$$U^0_\alpha p_\beta - U^0_\beta p_\alpha + s_{\mu\nu}(\Omega_{\mu\sigma} U^\sigma_\alpha U^\nu_\beta + U^\mu_\alpha \Omega_{\nu\sigma} U^\sigma_\beta) = 0 ,$$

$$\Omega_{\mu\sigma} s_{\mu\nu} U^\sigma_\alpha U^\nu_\beta - \Omega_{\nu\sigma} s_{\nu\mu} U^\sigma_\beta U^\mu_\alpha = a_\mu (U^\mu_\alpha U^0_\beta - U^\mu_\beta U^0_\alpha) .$$

Multiplying by $U^\chi_\alpha U^\theta_\mu$ and summing α and β we get

(6.6) $$\Omega_{\mu\chi} s_{\mu\theta} - \Omega_{\nu\theta} s_{\nu\chi} = a_\chi \delta_{0\theta} - a_\theta \delta_{0\chi} ,$$

iii)   a) $(x_\alpha, x_\beta) = (p_\alpha, p_\beta) = 0$.

These follow automatically from the definition of the P.b. For example, since

$$(6.7) \qquad m(U^\mu{}_\alpha, \; U^\nu{}_\beta) = \Omega_{\sigma\tau} \frac{\partial U^\mu{}_\alpha}{\partial U^\sigma{}_\pi} \frac{\partial U^\nu{}_\beta}{\partial U^\tau{}_\pi} = \Omega_{\mu\varrho} \delta_{\alpha\beta} \,,$$

then

$$m(p_\alpha, \; p_\beta) = m \, a_\mu a_\nu (U^\mu{}_\alpha, \; U^\nu{}_\alpha) = a_\mu a_\nu \Omega_{\mu\nu} \delta_{\alpha\beta} = 0 \,,$$

because of the asymmetry of $\Omega_{\mu\nu}$.

b) $\quad (x_\alpha, \; x_\beta) = \delta_{\alpha\beta} \,.$

This merely establishes the relation

$$(6.8) \qquad\qquad a_0 = m \,,$$

where m is the constant appearing in the definition of H, and thus has the dimensions of mass.

iv) $\quad (M_{\alpha\beta}, \; M_{\mu\nu}) = M_{\alpha\mu} \delta_{\beta\nu} + M_{\beta\nu} \delta_{\alpha\mu} - M_{\alpha\nu} \delta_{\beta\mu} - M_{\beta\mu} \delta_{\alpha\nu} \,.$

Because of eq. (6.3) the above is automatically satisfied for $M_{\alpha\beta} = L_{\alpha\beta}$. Therefore it reduces to the condition

$$s_{\lambda\tau} s_{\varrho\sigma} (U^\lambda{}_\alpha U^\tau{}_\beta, \; U^\varrho{}_\mu U^\sigma{}_\nu) = s_{\varrho\sigma} (U^\varrho{}_\mu U^\sigma{}_\nu \delta_{\alpha\beta} + U^\varrho{}_\beta U^\sigma{}_\nu \delta_{\alpha\mu} - U^\varrho{}_\alpha U^\sigma{}_\nu \delta_{\beta\mu} - U^\varrho{}_\beta U^\sigma{}_\mu \delta_{\alpha\nu}) \,.$$

Using (6.7) and equating the coefficients of the δ's appearing we get in each case the condition

$$(6.9) \qquad\qquad s_{\tau\lambda} \Omega_{\lambda\varrho} s_{\varrho\sigma} = - \, m s_{\tau\sigma} \,.$$

Eq. (6.5), (6.6), (6.8), (6.9) are sufficient to determine te a_μ, $s_{\mu\nu}$ in terms of the $\Omega_{\mu\nu}$ and m,

This will be demonstrated more easily if we consider $\Omega_{\mu\nu}$, $s_{\mu\nu}$, a_μ—which formally appear as 4-tensors—as two six-vectors and a scalar plus a 3-vector, respectively.

Let

$$(6.10) \qquad \begin{cases} (a) & a_\mu = [a_0, \; \boldsymbol{a}] \\[2mm] (b) & \Omega_{\mu\nu} = [\boldsymbol{\omega}; \; \boldsymbol{v}] \\[2mm] (c) & s_{\mu\nu} = [\boldsymbol{l}; \; \boldsymbol{p}] \end{cases}$$

so that

(6.11)
$$\begin{cases} \Omega_{0i} = \omega_i \\ \Omega_{jk} = \varepsilon_{ijk} v_i \end{cases}$$

(6.12)
$$\begin{cases} s_{0i} = l_i \\ s_{jk} = \varepsilon_{ijk} q_i \end{cases}$$

where Latin suffixes have the values $(1, 2, 3)$, and ε_{ijk} is the completely anti-symmetric tensor of rank 3.

(6.5) is now equivalent to the equation

(6.13)
$$\boldsymbol{a} \times \boldsymbol{v} = a_0 \boldsymbol{\omega} .$$

Eq. (6.5) is satisfied identically when $\chi = \theta = 0$. When $\theta = 0$, $\chi = i$ it becomes

(6.14)
$$\boldsymbol{q} \times \boldsymbol{\omega} + \boldsymbol{l} \times \boldsymbol{v} = \boldsymbol{a}$$

and when $\theta = i$, $\chi = j$ $(\neq i)$

(6.15)
$$\boldsymbol{l} \times \boldsymbol{\omega} + \boldsymbol{q} \times \boldsymbol{v} = 0 .$$

Eq. (6.9) is satisfied identically when $\tau = \sigma$. When $\tau = 0$, $\sigma = k$ it becomes

(6.16)
$$(\boldsymbol{l} \cdot \boldsymbol{\omega} + \boldsymbol{q} \cdot \boldsymbol{v}) \boldsymbol{l} - (\boldsymbol{l} \cdot \boldsymbol{q}) \boldsymbol{v} = m \boldsymbol{l}$$

and when $\tau = i$, $\sigma = j$ $(\neq k)$

(6.17)
$$(\boldsymbol{l} \cdot \boldsymbol{\omega} + \boldsymbol{q} \cdot \boldsymbol{v}) \boldsymbol{q} - (\boldsymbol{l} \cdot \boldsymbol{q}) \boldsymbol{\omega} = m \boldsymbol{q} .$$

We thus have the five equations:

$$\boldsymbol{a} \times \boldsymbol{v} = a_0 \boldsymbol{\omega} ,$$
$$\boldsymbol{q} \times \boldsymbol{\omega} + \boldsymbol{l} \times \boldsymbol{v} = \boldsymbol{a} ,$$
$$\boldsymbol{l} \times \boldsymbol{\omega} + \boldsymbol{q} \times \boldsymbol{v} = 0 ,$$
$$\lambda \boldsymbol{l} + \mu \boldsymbol{v} = m \boldsymbol{l} ,$$
$$\lambda \boldsymbol{q} + \mu \boldsymbol{\omega} = m \boldsymbol{q} ,$$

where

$$\lambda = \boldsymbol{l} \cdot \boldsymbol{\omega} + \boldsymbol{q} \cdot \boldsymbol{v} ,$$
$$\mu = \boldsymbol{l} \cdot \boldsymbol{q} ,$$

and

$$a_0 = m .$$

These reduce to the set

(6.18) $\boldsymbol{\omega} \cdot \boldsymbol{v} = 0 ,$

(6.19) $\boldsymbol{a} = \dfrac{a_0}{r^2} \, \boldsymbol{v} \times \boldsymbol{\omega} ,$

(6.20) $\boldsymbol{l} = 0 ,$

(6.21) $\boldsymbol{q} = \dfrac{a_0}{r^2} \, \boldsymbol{v} ,$

which, together with

$$a_0 = m ,$$

define a_μ, $s_{\mu\nu}$ in terms of m and $\Omega_{\mu\nu}$. We summarize the results:
With

$$\Omega_{0i} = \omega_i , \qquad \Omega_{ij} = \varepsilon_{ijk} v_k ,$$

so that

(6.22) $\omega^2 = \Omega_{0i} \Omega_{0i} , \qquad v^2 = \tfrac{1}{2} \Omega_{mn} \Omega_{mn} \qquad \omega^2 + v^2 = \tfrac{1}{2} \Omega_{\mu\nu} \Omega_{\mu\nu},$

the momentum and spin angular momentum of the particle are, respectively,

$$p_\alpha = m U^0{}_\alpha + a_i U^i{}_\alpha$$

and

$$S_{\alpha\beta} = s_{\mu\nu} U^\mu{}_\alpha U^\nu{}_\beta ,$$

where

(6.23) $a_i = \dfrac{m}{v^2} \, \Omega_{0\mu} \Omega_{\mu i} ,$

(6.24) $\begin{cases} s_{0i} = 0 , \\[2mm] s_{jk} = \dfrac{m}{v^2} \, \Omega_{jk} . \end{cases}$

Then $p_\alpha p_\alpha = m^2 + a^2$. To evaluate $a^2 = a_i a_i$ we get from (6.19)

(6.25a) $a^2 = \dfrac{m^2}{v^4} \, (\boldsymbol{v} \times \boldsymbol{\omega}) \cdot (\boldsymbol{v} \times \boldsymbol{\omega}) = \dfrac{m^2}{v^4} \, [v^2 \omega^2 - (\boldsymbol{v} \cdot \boldsymbol{\omega})^2] = m^2 \, \dfrac{\omega^2}{v^2} .$

Thus

(6.25b)
$$p_\alpha p_\alpha = m^2 \left(1 + \frac{\omega^2}{c^2}\right) = m^2 \frac{\lambda^2}{c^2} .$$

Also

$$\tfrac{1}{2} S_{\alpha\beta} S_{\alpha\beta} = \tfrac{1}{2} s_{\mu\nu} s_{\mu\nu} = \tfrac{1}{2} s_{ij} s_{ij} ,$$

but

(6.26)
$$\tfrac{1}{2} s_{ij} s_{ij} = \frac{1}{2} \frac{m^2}{c^4} \Omega_{ij} \Omega_{ij} = \frac{m^2}{c^2} ,$$

whence

$$\frac{m^2}{c^2} = \frac{1}{2} S_{ij} S_{ij} = \frac{1}{2} s_{\alpha\beta} s_{\alpha\beta} .$$

Eq. (6.18) is a relation to be satisfied by the $\Omega_{\mu\nu}$

(6.27)
$$\varepsilon_{ijk} \Omega_{0i} \Omega_{jk} = 0 .$$

It can be proved (see Appendix) that the $\Omega_{\mu\nu}$ satisfy also the relation

(6.28)
$$\Omega_{\alpha\mu} \Omega_{\mu\nu} \Omega_{\nu\beta} = - \lambda^2 \Omega_{\alpha\beta} ,$$

where $\lambda^2 = \omega^2 - c^2 = \tfrac{1}{2} \Omega_{\alpha\beta} \Omega_{\alpha\beta}$. The $s_{\mu\nu}$ satisfy a similar relation

(6.29)
$$s_{\alpha\mu} s_{\mu\nu} s_{\nu\beta} = - \frac{m^2}{c^2} s_{\alpha\beta} .$$

From (6.29) follows that the $S_{\alpha\beta}$ satisfy

(6.30)
$$S_{\alpha\mu} S_{\mu\nu} S_{\nu\beta} = - \tfrac{1}{2} (S_{\sigma\tau} S_{\sigma\tau}) S_{\alpha\beta} .$$

7. – The equation of motion for the point P.

Substituting the values for the $a_{\mu\nu}$ given by (6.23), in (5.1) we have

$$p_\alpha = m U^0_\alpha + \frac{m}{c^2} \Omega_{0\mu} \Omega_{\mu i} U^i_\alpha ,$$

$$= m U^0_\alpha + \frac{m}{c^2} (\Omega_{0\mu} \Omega_{\mu\nu} U^\nu_\alpha - \Omega_{0\mu} \Omega_{\mu 0} U^0_\alpha) ,$$

$$= m \left(1 + \frac{\omega^2}{c^2}\right) U^0_\alpha + \frac{m}{c^2} \Omega_{0\mu} \dot{U}^\mu_\alpha ,$$

where $\text{dot} \equiv \mathrm{d}/\mathrm{d}\xi$. Since the $U^{\mu}{}_{\wedge}$ do not depend explicitly on ξ it follows from (4.11) that

$$\Omega_{\sigma\mu}\dot{U}^{\mu}{}_{\alpha} = \ddot{U}^{\sigma}{}_{\alpha}.$$

Hence

(7.1)
$$p_{\alpha} = m\,\frac{\lambda^2}{v^2}\,U^0{}_{\alpha} + \frac{m}{v^2}\,\ddot{U}^0{}_{\alpha}.$$

Eq. (7.1) is a second order differential equation for $U^0{}_{\alpha}$. Since $U^0{}_{\alpha} = \dot{x}_{\alpha}$, where x_{ν} are the co-ordinates of P, it can be regarded as the equation of motion for P.

For the free particle p_{α} is constant and eq. (7.1) is easily solved giving

(7.2)
$$U^0{}_{\alpha} = A_{\alpha} \cos \lambda\xi + B_{\alpha} \sin \lambda\xi + \frac{p_{\alpha}}{m}\left(\frac{v^2}{\lambda^2}\right),$$

where A_{α}, B_{α} are arbitrary constant vectors.

This shows that the motion of the point P consists of an oscillatory motion superimposed on a constant rectilinear motion. The constant rectilinear part is proportional to the total momentum and is

$$\frac{m\,p_{\alpha}}{p_{\mu}\,p_{\mu}} = \left(\frac{m}{M}\right)\frac{p_{\alpha}}{M},$$

where $M = \sqrt{p_{\mu}p_{\mu}} = c$ times the proper mass of the particle. The oscillatory part has a frequency λ where $\lambda^2 = p_{\alpha}p_{\alpha}/\frac{1}{2}S_{\mu\nu}S_{\mu\nu}$ which will be very large if the spin is small compared to the momentum. Eq. (7.2) is similar to Dirac's solution for the motion of the free electron [8].

The result (7.2) is a particular case of the application of equation (6.28) to the equations of motion (3.3) for the $U^{\mu}{}_{\alpha}$. Multiplying (6.28) by $U^{\sigma}{}_{\beta}$ and summing over β we see that each of the $U^{\mu}{}_{\alpha}$ satisfies the equation

$$\Omega_{\alpha\mu}\,\Omega_{\mu\nu}\,\Omega_{\nu\beta}\,U^{\sigma}{}_{\beta} = -\lambda^2\Omega_{\alpha\beta}U^{\sigma}{}_{\beta}$$

which can be written as the differential equation

(7.3)
$$(D^3 + \lambda^2 D)U^{\mu}{}_{\alpha} = 0,$$

where $D \equiv \mathrm{d}/\mathrm{d}\xi$.

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[8] P. A. M. DIRAC: *The Principles of Quantum Mechanics* (Oxford, 1947), § 69.

Hence each of the $U^\mu_{\ \alpha}$ has the form

(7.4) $$U^\mu_{\ \alpha} = A^\mu_{\ \alpha} \cos \lambda\xi + B^\mu_{\ \alpha} \sin \lambda\xi + C^\mu_{\ \alpha},$$

where $A^\mu_{\ \alpha}$, $B^\mu_{\ \alpha}$, $C^\mu_{\ \alpha}$ are arbitrary constant vectors.

8. – Comparison with the Weyssenhoff and the Bohm-Vigier theories.

In this theory as in the Weyssenhoff and Bohm-Vigier theories the motion of the free particle is described in terms of a velocity, a momentum, and a spin angular momentum, for which a) the momentum is not parallel to the velocity, and b) the spin angular momentum satisfies an equation which in our notation is

(8.1) $$S_{\alpha\beta} = U^0_{\ \beta}p_\alpha - U^0_{\ \alpha}p_\beta.$$

WEYSSENHOFF assumed as well that the inner product of the spin angular momentum and the velocity is zero. BOHM and VIGIER showed that for a particle, for which a) and b) above are satisfied, an additional assumption is necessary to define the motion completely. Such an assumption could be the one assumed by WEYSSENHOFF. Thus Weyssenhoff's theory is a particular case of the Bohm-Vigier theory representing one possible motion for the particle. But in the latter theory other motions are possible depending on the assumption made to close the system.

It follows from (5.3) and (6.24) that in our theory

(8.2) $$S_{\alpha\beta}U^0_{\ \alpha} = \frac{m^2}{v^2}\,\Omega_{jk}U^j_{\ \alpha}U^k_{\ \beta}U^0_{\ \alpha} = 0.$$

Thus Weyssenhoff's condition is satisfied. It is not, however, an *a priori* assumption but results from our choice of the quantities $s_{\mu\nu}$ since we took $s_{0i} = 0$. The $s_{\mu\nu}$ we determined by equations (6.1) to (6.4). Of these eq. (6.1) and (6.2) occur also in the Weyssenhoff and Bohm-Vigier theories and (6.3) gave no conditions on the $s_{\mu\nu}$. Thus (8.2) results essentially because of the assumption regarding the P.b. (eq. (6.4)) of the components of the total angular momentum tensor.

Consider now the pseudovector s_μ defined by

(8.3) $$s_\mu = \tfrac{1}{2}\varepsilon_{\mu\nu\alpha\beta}S_{\alpha\beta}U^0_{\ \nu} = S'_{\mu\nu}U^0_{\ \nu},$$

where $S'_{\mu\nu}$ is the dual tensor of $S_{\mu\nu}$. Obviously $s_\mu U^0_{\ \mu} = 0$. Also

(8.4) $$s_\mu s_\mu = \tfrac{1}{2}S_{\alpha\beta}S_{\alpha\beta},$$

that is, s_μ has the same magnitude as $S_{\lambda\beta}$. It is also constant (as in the other theories).

From (6.24)

$$s_\mu = \tfrac{1}{2}\varepsilon_{\mu\nu\alpha\beta}s_{ij}U^i{}_\alpha U^j{}_\beta U^0{}_\nu .$$

The quantity $\varepsilon_{\mu\nu\alpha\beta}U^i{}_\alpha U^j{}_\beta U^0{}_\nu$ is obviously orthogonal to $U^i{}_\mu$, $U^j{}_\mu$, $U^0{}_\mu$, and hence we can write

(8.5) $$\varepsilon_{\mu\nu\alpha\beta}U^i{}_\alpha U^j{}_\beta U^0{}_\nu = \varepsilon_{0ijk}U^k{}_\mu .$$

This gives

(8.6) $$s_\mu = \tfrac{1}{2}s_{ij}\varepsilon_{0ijk}U^k{}_\mu ,$$

(8.7) $$= \frac{m}{2v^2}\varepsilon_{ijk}\Omega_{ij}U^k{}_\mu = \frac{m}{2v^2}v_k U^k{}_\mu ,$$

Then

$$s_\mu = \frac{m}{2v^2}\varepsilon_{ijk}\Omega_{ij}\Omega_{\lambda\tau}U^\tau{}_\mu ,$$

$$= \frac{m}{2v^2}\left[\varepsilon_{ijk}\Omega_{ij}\Omega_{k0}U^0{}_\mu + \varepsilon_{ijk}\Omega_{ij}\Omega_{kl}U^l{}_\mu\right].$$

The first term is zero on account of (6.27) and the second since

(8.8) $$\varepsilon_{ijk}\Omega_{ij}\Omega_{kl} = \tfrac{1}{2}v_k\Omega_{kl} = \tfrac{1}{2}v_k\varepsilon_{klm}v_m = 0 .$$

Thus s_μ is constant. Following Bohm-Vigier we call it the spin pseudovector.

For future reference we note that the inner product of $S_{\alpha\beta}$ with its dual $S'_{\alpha\beta}$ is zero,

$$S_{\alpha\beta}S'_{\beta\gamma} = \frac{m^2}{2v^4}\Omega_{ij}\Omega_{lm}\varepsilon_{\beta\gamma\mu\nu}U^i{}_\alpha U^j{}_\beta U^l{}_\mu U^m{}_\nu ,$$

$$= \frac{m^2}{2v^4}\Omega_{ij}\Omega_{lm}\varepsilon_{jlm}U^i{}_\alpha U^0{}_\gamma \ (^9) ,$$

which is zero on account of (8.8). Hence

(8.9) $$S_{\alpha\beta}S'_{\beta\gamma} = 0$$

and in particular

(8.10) $$S_{\alpha\beta}S'_{\alpha\beta} = 0 .$$

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(9) $\varepsilon_{\beta\gamma\mu\nu}U^j{}_\beta U^l{}_\mu U^m{}_\nu = \varepsilon_{jlmk}U^k{}_\gamma$ see (8.5) ,

$\qquad\qquad = \varepsilon_{jlm}U^0{}_\gamma$ (since $j, l, m, \neq 0$) .

9. – The Frenet tetrad.

If we write

$$p_\alpha = m U^0{}_\alpha + r_\lambda ,$$

then

(9.1) $$r_\alpha = a_i U^i{}_\alpha$$

and

(9.2) $$r_\alpha U^0{}_\alpha = 0 .$$

With the vectors $U^0{}_\lambda$, r_λ we can form with $S_{\alpha\beta}$ the following tensor and pseudo-tensor quantities

(9.3a) $$S_{\alpha\beta} U^0{}_\alpha , \qquad S'_{\alpha\beta} U^0{}_\beta ,$$

(9.3b) $$S_{\alpha\beta} r_\alpha , \qquad S'_{\alpha\beta} r_\beta .$$

Of the quantities (9.3a) the first is identically zero, and the other is the spin pseudovector s_μ

$$s_\alpha = S'_{\alpha\beta} U^0{}_\beta = \frac{m}{2v^2} v_k U^k{}_\alpha .$$

Of the quantities (9.36) the second is identically zero

$$S'_{\alpha\beta} r_\beta = \frac{1}{2v^2} \Omega_{jk} a_i \varepsilon_{\alpha\beta\mu\nu} U^j{}_\mu U^k{}_\nu U^i{}_\beta ,$$

$$= \frac{1}{2v^2} \varepsilon_{ijk} \Omega_{ik} a_i U^0{}_\alpha ,$$

$$= \frac{1}{2v^2} a_i v_i U^0{}_\alpha = 0 ,$$

and the other is a quantity which we shall call q_α

(9.4) $$q_\alpha = S_{\alpha\beta} r_\beta = \frac{1}{v^2} \Omega_{jk} a_j U^k{}_\alpha ,$$

$$= \frac{1}{v^2} \varepsilon_{jkl} v_l a_j U^k{}_\alpha = -\frac{m}{v^2} \omega_i U^i{}_\alpha .$$

8 - *Supplemento al Nuovo Cimento.*

The four quantities U^0_α, q_α, r_α, s_α are mutually orthogonal

(9.5)
$$\begin{cases} U^0_\alpha q_\alpha = S_{\alpha\beta} U^0_\alpha r_\beta = 0\,, \\ U^0_\alpha r_\alpha = 0\,, \qquad U^0_\alpha s_\alpha = 0\,, \end{cases}$$

(9.6)
$$\begin{cases} q_\alpha r_\alpha = S_{\alpha\beta} r_\alpha r_\beta = 0\,, \\ q_\alpha s_\alpha = S_{\alpha\beta} r_\beta S'_{\alpha\gamma} U^0_{\ \gamma} = 0\,, \end{cases}$$

(9.7)
$$r_\alpha s_\alpha = a_i U^i_\alpha s_\alpha = \frac{m}{2v^2}\, a_i v_k U^i_\alpha U^k_\alpha = \frac{m}{2v^2}\, a_k v_k = 0\,.$$

Let V^0_α, V^1_α, V^2_α, V^3_α be unit vectors in the directions of U^0_α, $-q_\alpha$, r_α, s_α respectively. Then

(9.8)
$$\begin{cases} V^0_\alpha = U^0_\alpha\,, \\[4pt] V^1_\alpha = \dfrac{\omega_i U^i_\alpha}{\sqrt{\omega^2}}\,, \\[8pt] V^2_\alpha = \dfrac{a_i U^i_\alpha}{\sqrt{a^2}}\,, \\[8pt] V^3_\alpha = \dfrac{v_i U^i_\alpha}{\sqrt{v^2}}\,, \end{cases}$$

and

$$\dot V^0_\alpha = \Omega_{0\mu} U^\mu_\alpha = \Omega_{0i} U^i_\alpha = \omega_i U^i_\alpha = \sqrt{\omega^2}\, V^1_\alpha\,,$$

$$\dot V^1_\alpha = \frac{1}{\sqrt{\omega^2}}\, \omega_i \Omega_{i\mu} U^\mu_\alpha = \frac{1}{\sqrt{\omega^2}}\, (\omega_i \varepsilon_{ijk} v_k U^j_\alpha + \omega_i \Omega_{i0} U^0_\alpha) = -\sqrt{\omega^2}\, V^0_\alpha + \sqrt{v^2} V^2_\alpha\,,$$

using (6.25a)

$$\dot V^2_\alpha = \frac{1}{\sqrt{a^2}}\, a_i \Omega_{i\mu} U^\mu_\alpha = \frac{a_i}{\sqrt{a^2}}\, (\varepsilon_{ijk} v_k U^j_\alpha + \omega_i U^0_\alpha) = -\sqrt{v^2}\, V^1_\alpha\,,$$

$$\dot V^3_\alpha = 0$$

(since V^3_α is parallel to s_α).

Introducing quantities ϱ and σ defined by

(9.9)
$$\frac{1}{\varrho^2} = \omega^2\,, \qquad \frac{1}{\sigma^2} = v^2\,,$$

the equations for the derivatives of the V^μ_α are

$$(9.10) \quad \begin{cases} \dot{V}^0_\alpha = \dfrac{1}{\varrho} V^1_\alpha, \\[2ex] \dot{V}^1_\alpha = -\dfrac{1}{\varrho} V^0_\alpha \qquad\qquad + \dfrac{1}{\sigma} V^2_\alpha, \\[2ex] \dot{V}^2_\alpha = \qquad\qquad -\dfrac{1}{\sigma} V^1_\alpha, \\[2ex] \dot{V}^3_\alpha = \qquad\qquad 0, \end{cases}$$

which are the Frenet equations in 4-space [10] for the direction cosines of an orthogonal tetrad of vectors moving along a curve which has a constant trinormal, that is, zero tilt. V^1_α, V^2_α, V^3_α are the direction-cosines of the normal (or principal normal), binormal, and trinormal to the curve, respectively. $1/\varrho$ is the curvature; $1/\sigma$ the tilt of the curve. We shall refer to the orthogonal frame specified by the direction-cosines V^μ_α as the Frenet-frame.

Since they are orthogonal unit vectors the V^μ_α satisfy

$$(9.11) \qquad\qquad V^\mu_\alpha V^\nu_\alpha = \delta^{\mu\nu}; \qquad V^\mu_\alpha V^\mu_\beta = \delta_{\alpha\beta}.$$

Moreover, since they are linear combinations of the U^μ_α we can write

$$(9.12) \qquad\qquad V^\mu_\alpha = O_{\mu\nu} U^\nu_\alpha.$$

Regarding V^μ_α (for μ varying, α fixed) as a column vector V_α, (9.12) can be written as the matrix equations

$$(9.13) \qquad\qquad V_\alpha = O U_\alpha,$$

where, because of the orthogonality relations (3.1), (9.11), O must be an orthogonal matrix

$$(9.14) \qquad\qquad O\tilde{O} = \tilde{O}O = 1.$$

(This is easily verified using eq. (9.8), (6.18), (6.19).)

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[10] See A. R. FORSYTH: *Geometry of Four Dimensions* (Cambridge, 1930), § 164 whose terminology we use.

10. – Invariance of the dynamics scheme.

Our basic equations are

i) the definition of the P.b.

$$(u, v) - \frac{\partial u}{\partial x_\alpha} \frac{\partial v}{\partial U^0_\alpha} - \frac{\partial u}{\partial U^0_\alpha} \frac{\partial v}{\partial x_\alpha} + \Omega_{\mu\nu} \frac{\partial u}{\partial U^\mu_\alpha} \frac{\partial v}{\partial U^\nu_\alpha} ,$$

and ii) the definition of the Hamiltonian

$$H = \tfrac{1}{2} m U^\mu{}_\alpha U^\mu{}_\alpha .$$

It is easy to show that for transformations of the type (9.12) these equations retain their form.

Eq. (9.12) can be written

(10.1) $$U^\nu{}_\alpha = O_{\mu\nu} V^\mu{}_\alpha ,$$

so that

$$\frac{\partial U^\nu{}_\alpha}{\partial V^\sigma{}_\beta} = O_{\mu\nu} \frac{\partial V^\mu{}_\alpha}{\partial V^\sigma{}_\beta} = O_{\sigma\nu} \delta_{\alpha\beta} .$$

Hence

$$\frac{\partial u}{\partial V^\mu{}_\alpha} = \frac{\partial u}{\partial U^\nu{}_\beta} \frac{\partial U^\mu{}_\beta}{\partial V^\mu{}_\alpha} = O_{\mu\nu} \frac{\partial u}{\partial U^\nu{}_\alpha} ,$$

or

(10.2) $$\frac{\partial u}{\partial U^\sigma{}_\alpha} = O_{\mu\sigma} \frac{\partial u}{\partial V^\mu{}_\alpha} .$$

From (10.2) it follows that the P.b. can be written

(10.3) $$(u, v) = \frac{\partial u}{\partial x_\alpha} \frac{\partial v}{\partial V^\sigma{}_\alpha} - \frac{\partial u}{\partial V^0{}_\alpha} \frac{\partial v}{\partial x_\alpha} + \Omega'_{\mu\nu} \frac{\partial u}{\partial V^\mu{}_\alpha} \frac{\partial v}{\partial V^\nu{}_\alpha} ,$$

where

(10.4) $$\Omega'_{\mu\nu} = O_{\mu\sigma} \Omega_{\sigma\tau} O_{\nu\tau}$$

or in matrix notation

(10.5) $$\Omega' = O \Omega \tilde{O}$$

and from (10.1)

(10.6)
$$
\begin{cases}
H = \dfrac{m}{2}\, U^{\mu}{}_{\alpha} U^{\mu}{}_{\alpha} = \dfrac{m}{2}\, V^{\mu}{}_{\alpha} V^{\mu}{}_{\alpha}\,, \\[2ex]
\dot{V}^{\mu}{}_{\alpha} = \Omega'_{\mu\nu} V^{\nu}{}_{\alpha}\,.
\end{cases}
$$

Thus the dynamics scheme is invariant to an orthogonal transformation of the moving frame or reference. Consequently the motion could be discussed in terms of the Frenet frame ([11]). It will be seen from (9.10) that by doing so our equations will be simplified since then only four of the quantities $\Omega'_{\mu\nu}$ will be non-zero.

11. – Concluding remarks.

In this paper a generalized Hamiltonian dynamics for classical finite particles with spin has been introduced. The main assumptions made are:

i) that the motion of the particle can be described in terms of the position co-ordinates of some point P within the body, and the direction-cosines of an orthogonal tetrad of vectors forming a moving frame of reference whose origin is P;

ii) that the motion of the particle can be described in terms of its momentum, its external angular momentum, and its spin angular momentum.

Accepting these assumptions it was found that the Frenet frame could be used as the moving frame of reference.

In a subsequent paper the motion will be discussed relative to the Frenet frame. This results in great simplification since only four of the quantities $\Omega_{\mu\nu}$ will be non-zero. The motion will also be discussed in terms of a frame obtained from the Frenet frame by a rotation leaving V^1 and V^3 unchanged. The resulting scheme has a quantum analogue yielding a wave equation in which the mass term includes an operator involving internal co-ordinates of the particle only, thus introducing the possibility of mass quantization.

([11]) Use of the Frenet equations in obtaining spinor equation of motion for classical point particles was made by GÜRSEY. F. GÜRSEY: *Nuovo Cimento*, **5**, 784 (1957).

APPENDIX

Proof of Relation (6.28).

First of all we note that because of (6.24) and (6.11)

$$\Omega_{i0}\Omega_{0l}\Omega_{lj} + \Omega_{ik}\Omega_{k0}\Omega_{0j} = \frac{v^2}{a_0}\left(-\,\omega_i a_j + a_i \omega_j\right),$$

$$= \frac{v^2}{a_0}\left[\boldsymbol{a}\times\boldsymbol{\omega}\right]_k, \qquad\qquad (k \neq i,\,j),$$

$$= \left[(\boldsymbol{v}\times\boldsymbol{\omega}) + \boldsymbol{\omega}\right]_k = -\,\omega^2 v_k.$$

Hence

(A-1) $\Omega_{i0}\Omega_{0l}\Omega_{lj} + \Omega_{ik}\Omega_{k0}\Omega_{0j} = -\,\omega^2 \Omega_{ij}.$

From (6.19)

$$s_{\tau\lambda}s_{\lambda\varrho}s_{\varrho\sigma} = -\,m s_{\tau\sigma},$$

but since $s_{i0} = 0$

$$s_{ik}\Omega_{kl}s_{lj} = -\,m s_{ij}.$$

Substituting for s_{ik} from (6.23)

$$\frac{a_0}{v^2}\,\Omega_{ik}\Omega_{kl}\Omega_{lj}\,\frac{a_0}{v^2} = m\Omega_{ij}\,\frac{a_0}{v^2}.$$

This result together with (A-1) gives

$$\Omega_{ik}\Omega_{kl}\Omega_{lj} + \Omega_{i0}\Omega_{0l}\Omega_{lj} + \Omega_{ik}\Omega_{k0}\Omega_{0j} = -\,v^2\Omega_{ij} - \omega^2\Omega_{ij},$$

that is,

(A-2) $\Omega_{i\mu}\Omega_{\mu\nu}\Omega_{\nu j} = -\,(\omega^2 + v^2)\Omega_{ij}.$

Now consider

(A-3) $\Omega_{i\mu}\Omega_{\mu\nu}\Omega_{\nu 0} = \Omega_{i0}\Omega_{0\nu}\Omega_{\nu 0} + \Omega_{ij}\Omega_{j\nu}\Omega_{\nu 0},$

$$= -\,\omega^2\Omega_{i0} + \varepsilon_{jli}\varepsilon_{jkm}v_l v_m\Omega_{k0},$$

$$= -\,\omega^2\Omega_{i0} - v_k v_i\omega_i - v^2\Omega_{i0},$$

$$= -\,(v^2 + \omega^2)\Omega_{i0}.$$

Therefore from (A-2) and (A-3) it follows that

(A-4) $\Omega_{\alpha\mu}\Omega_{\mu\nu}\Omega_{\nu\beta} = -\,(\omega^2 + v^2)\Omega_{\alpha\beta} = -\,\lambda^2\Omega_{\alpha\beta} = -\,(\tfrac{1}{2}\,\Omega_{\mu\nu}\Omega_{\mu\nu})\Omega_{\alpha\beta}.$

A similar relation holds for the $s_{\mu\nu}$:

(A-5) $$s_{\alpha\mu}s_{\mu\nu}s_{\nu\beta} = - \left(\tfrac{1}{2} s_{\mu\nu}s_{\mu\nu}\right)s_{\alpha\beta} \ .$$

Since $s_{0i} = 0$ it is obviously true when either $\alpha = 0$ or $\beta = 0$. Otherwise the left-hand side is

$$s_{i\mu}s_{\mu\nu}s_{\nu j} = s_{ik}s_{kl}s_{lj} \ ,$$

by (6.23) $$= \frac{a_0}{v^2} s_{ik}\Omega_{kl}s_{lj} \ ,$$

by (6.9) $$= \frac{a_0}{v^2} (- a_0 s_{ij}) \ ,$$

$$= - \frac{m^2}{v^2} s_{ij} = - \left(\tfrac{1}{2} s_{\mu\nu}s_{\mu\nu}\right)s_{ij} \ ,$$

Hence (A-5) is established.

Further Experiments with PAPA.

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Contents: Introduction. – **1.** The checkerboard experiment. – **2.** The music experiment.

Introduction.

The PAPA machine, built in our Institute, has been described elsewhere, and its ability to recognize simple geometrical patterns has been demonstrated [1]. In the experiments reported here we wanted to test the ability of PAPA in recognizing more sophisticated « cultural » patterns. By this term we mean patterns whose significance is given by « hidden rules » which are not obviously apparent to a superficial observer. For example, in experiment No. 1 (Section **1** below) we presented a checkerboard in which holes connected by rook moves and knight moves respectively were given to the machine for recognition; in experiment No. 2 (Section **2** below) we coded a few notes from Bach's chorals and from « fake » music made by one of us (L. Gamberini) and we asked the machine to learn these two classes of patterns.

The results turned out to be exceedingly encouraging and we think that this proves the « intelligence » of our machine. Given the appropriate development by increasing the number of association units, PAPA can be a decision making machine that, for instance, by looking at an electrical wiring diagram can decide whether it is a « good » one or not, given the pertinent instruction *by examples*. Or it can become a weather forecasting machine when shown examples of « good » and « bad » forecasting from a collection of physical data. In other words, PAPA will behave « as if » it had guessed the rules.

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[1] A. Gamba: *Proc. IRE*, **49**, 349 (1961); G. Palmieri and R. Sanna: *Methodos*, to be published.

1. – The checkerboard experiment.

In cards representing a checkerboard, such as in Fig. 1, we punched 12 holes. In the first class the 12 holes were connected continuously by rook moves and in the second class by knight moves. We punched a total of 300 cards (150 of each type) all different. One hundred cards of each type were shown to PAPA during the learning period and the remaining 50 of each type were kept apart to be shown later to the machine for recognition. Again we stress the fact that these were all different from each other and from those shown during the training period.

Fig. 1. – Rook moves (top) and knight moves (bottom) recognized by PAPA. The recognition factors (negative when wrong) refer to 12×79 equivalent A-units.
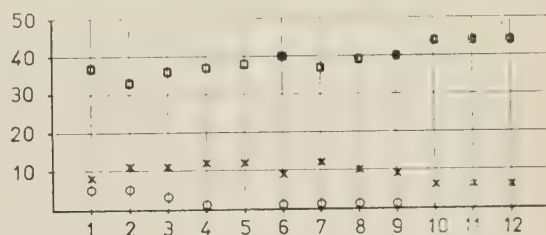
As described elsewhere ([1]) our PAPA has 79 A-units. In order to see how it would behave with a greater number of A-units, one could change the random masks in each A-unit and repeat the experiment, combining the answers with those obtained before. A more simple approach is to rotate the PAPA (or, equivalently, the patterns being shown) and consider it as another PAPA with another (rotated) 79 A-units. In our experiments we rotated the patterns by 30° each time and thus could repeat the learning curves as the number of A-units increased from 79 to 12×79. The results are summarized in Fig. 2 and 3 for the rook and knight moves,

Fig. 2. – Learning curves for rook moves as the number of A-units increases from 1×79 to 12×79. The abscissa gives the number of A-units as a multiple of 79. The ordinate gives the number of patterns recognized correctly (□), uncorrectly (×), and given even (○) out of the 50 examples presented.
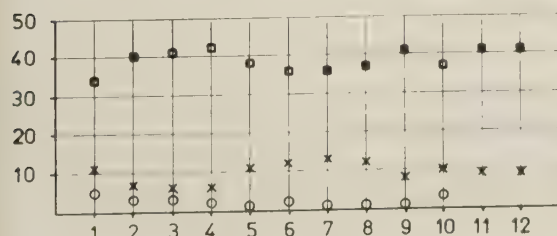

Fig. 3. – Learning curves for knight moves. Same notations as in Fig. 2.

respectively. The recognition factor (given as logarithms to base 10) for the equivalent 12 ✕ 79 A-units ranged from a maximum of 70 down to — 49 with an average of 22.70.

2. – The music experiment.

A few notes from various Bach chorals were coded into cards, so that each card corresponded to a simple musical phrase. The coding is more easily explained by giving two examples (see Fig. 4). The sign at the bottom right stands for the pause at the end of the tune. By such a device the total area of each phrase was then always the same. 225 tunes such as these—all in the same key— were selected by one of us (GAMBERINI).

Another 225 tunes for the other class were then prepared as examples of « fake » music. They were made so that they sound « bad » when played on the piano keyboard and compared with the « good » Bach music. They were, however, chosen so that it was not immediately apparent from the coded cards which music belonged to which author (see Fig. 5). We wanted to see whether PAPA was able to form the concept of « Bach music » vs. « non-Bach music », whatever this might mean.

Fig. 4. – Examples showing how music was coded for experiment no. 2.
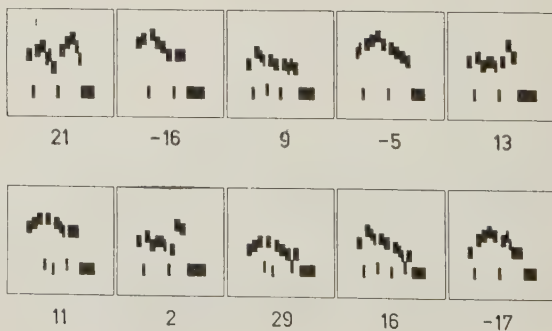
50 examples of each class were then selected at random

Fig. 5. – Examples of Bach music (bottom) and non-Bach music (top) recognized by PAPA. The recognition factors (negative when wrong) refer to 12 × 79 equivalent A-units.

and put aside to be shown to PAPA after instructing it with the remaining 175 examples.
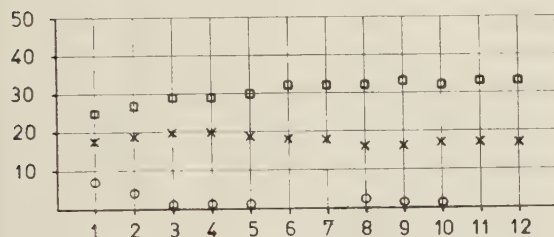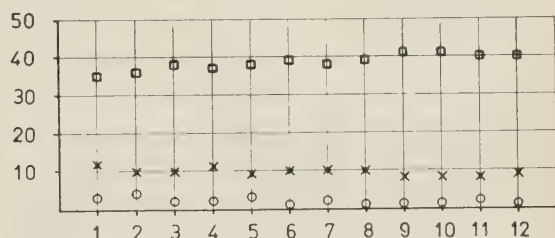
The results are given in Fig. 6 and 7.

Fig. 6. – Learning curves for Bach music. Same notations as in Fig. 2.

Fig. 7. – Learning curves for non-Bach music. Same notations as in Fig. 2.

After this experiment was completed we discovered that, on the average, the Bach music had a shorter pause at the end than non-Bach music. This has certainly introduced some bias, whose amount should not, however, substantially alter the result of our experiment. The recognition factors here ranged from a maximum of 71 down to — 57 with an average of 14.42.

Inelastic Collisions and Threshold Effects.

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1. – Introduction.

In a number of recent theoretical papers the many-channel problem has
been extensively discussed with particular emphasis on the anomalous energy
dependence which is observed in the scattering and reaction cross-sections at
the threshold of a new inelastic process ([1-19]). It is the purpose of this paper
to review such phenomena by introducing a new method which is based

(*) Present address.
([1]) E. P. WIGNER: *Phys. Rev.*, **73**, 1002 (1948).
([2]) G. BREIT: *Phys. Rev.*, **107**, 1612 (1957).
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([4]) A. I. BAZ: *Žurn. Èksp. Teor. Fiz.* **33**, 923 (1957); translation: *Soviet Phys. JETP*, **6**, 709 (1958).
([5]) R. G. NEWTON: *Ann. Phys.*, **4**, 29 (1958).
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([10]) L. M. DELVES: *Nucl. Phys.*, **9**, 391 (1958-59).

on a thorough investigation of the properties of the complete Green's function which describes the multi-channel system. The threshold behaviour of the Green's function which gives the propagation of probability in the newly opened channel, will prove to be the origin of the anomalous effects in the complete Green's function and consequently in the various elements of the transition amplitude. The method is able to describe both non-Coulomb and Coulomb effects. In the former case, anomalies of the kind of a cusp or of a rounded step show up in the old cross-sections at the threshold for the new inelastic process, due to the sudden removal of flux from the incident beam at the onset of the new inelastic cross-section starting with an infinite slope. On the contrary, in the latter case no cusp or rounded step appear; instead in the case of Coulomb attraction a finite jump is expected in the various old cross-sections due to the non-zero value of the new inelastic cross-section at its own threshold [7,9]. No anomaly is observed, however, at the threshold for a reaction leading to three (or more) particles, apart from an unobservable discontinuity in the second derivative of the old cross-sections. If consideration is given to reactions whose final three-body channels have at least one particle in common, a threshold effect then appears as a function of the momentum of the common particle with the total energy kept fixed [17-19].

To introduce the reader to the many-channel problem, we will discuss in Section 2 a simple example, already considered in ref. [5,20-22], which is an extension of the method of the distorted waves developed in the theory of atomic collisions [23]. In this way we will be able to define our formalism and to understand various features of the many-channel problem.

The general time-independent scattering theory which covers the case of

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[17] L. FONDA and R. G. NEWTON: Phys. Rev., 119, 1394 (1960).

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[21] M. CINI and S. FUBINI: Nuovo Cimento, 2, 75 (1955).

[22] H. FESHBACH: Ann. Phys., 5, 357 (1958).

[23] N. F. MOTT and H. S. W. MASSEY: Theory of Atomic Collisions, II Ed. (Oxford, 1949).

Coulomb forces and many-channels, is considered in detail in Section **3**. In Section **4** the low energy properties of scattering and reaction cross-sections are reviewed. The threshold effects are discussed in Sections **5** and **6**. A discussion on the importance of such phenomena for the determination of relative parities and spins of the reaction products, of scattering phase shifts, of cross-sections for processes which are often not feasible experimentally, and in the search for new particles, is given in Section **5**.

2. – The many-channel problem.

We will consider in this section the collision of a particle with a target capable of a certain number of excited states. The target particle will be supposed to be composed by A particles, while the incident particle will be taken, for simplicity, to have no internal structure. The interactions between the $A+1$ particles will be described by a set of potentials. In this example only the possibility for scattering (elastic or inelastic) of the incident particle on the target will be discussed. Also effects due to the identity of particles will not be taken into account. All the $A+1$ particles will be, for that matter, as different from one another. Such a description would be, therefore, an approximation to the actual situation of, for instance, collision of a nucleon with a nucleus since both the possibility of exchange scattering, of production of deuterons or α-particles, and the requirement of an antisymmetrical total wave function will not be considered.

Let $\mathscr{H}'$ be the total hamiltonian describing the system target+incident particle. We write for it

$$(2.1) \qquad \mathscr{H}' = T_0 + T_A^{\text{c.m.}} + H_A + V_{0A},$$

where H_A describes the target nucleus in its center of mass frame of reference, T_0 is the kinetic energy of the incoming nucleon, $T_A^{\text{c.m.}}$ is the kinetic energy of the center of mass of the nucleus. V_{0A} is the sum over all the elementary potentials between the incident particle and the A particles of the target nucleus.

Introducing co-ordinates $\boldsymbol{r}_0$, $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, ..., $\boldsymbol{r}_A$ measured, for convenience, from the center of mass of the system A, we separate the motion of the center of mass of the $A+1$ system and obtain the Schrödinger equation

$$(2.2) \qquad \mathscr{H}\Psi(\boldsymbol{r}_0, \boldsymbol{r}_1, ..., \boldsymbol{r}_A) = E\Psi(\boldsymbol{r}_0, \boldsymbol{r}_1, ..., \boldsymbol{r}_A),$$

with $\mathscr{H}$ given by

$$(2.3) \qquad \mathscr{H} = -\frac{\hbar^2}{2\mu_{0A}}\Delta_0 + H_A + V_{0A},$$

where $\mu_{0A} = m_0 m_A / (m_0 + m_A)$ is the reduced mass for the incident particle.

Let u_i be the complete set of eigenfunctions of H_A:

$$(2.4) \qquad H_A u_i(\mathbf{r}_1, ..., \mathbf{r}_A) = E_i u_i(\mathbf{r}_1, ..., \mathbf{r}_A) ,$$

H_A will have in general a discrete plus a continuous spectrum. The discrete spectrum represents the various excited states of the target nucleus which, for our collision problem, will characterize the « discrete or two-body channels ». The continuous spectrum represents the various possibilities for the nucleus to break-up, *i.e.* the realization for our collision problem of « continuous (three or more body) channels ». In the process of collision of the incident nucleon with the target nucleus in a certain state u_i, the potential V_{0A} will induce transitions between the various energy levels E_k of the nucleus and finally the incident nucleon will leave the nucleus in a state u_f. We will say that the reaction *channel i → channel f* has occurred. In so far as channel f is a two-body channel, the kinetic energies of the two final products will be completely specified once the total energy E is given (the initial channel i is a two-body channel for obvious experimental reasons). On the contrary, if channel f is a three or more body channel, we will have a continuous range of energies for the three or more products present in it for a given total energy E. The continuous channels will be neglected in the further developments of our example. This will be a good approximation if the energy E is far from the threshold for break-up of the nucleus.

In the absence of the exclusion principle, we can write for the total wave function Ψ the following expansion:

$$(2.5) \qquad \Psi(\mathbf{r}_0, \mathbf{r}_1, ..., \mathbf{r}_A) = \sum_i u_i(\mathbf{r}_1, ..., \mathbf{r}_A) \psi_i(\mathbf{r}_0) .$$

Introducing such an expansion for Ψ into eq. (2.2), we get the following set of coupled differential equations:

$$(2.6) \qquad (E - H_{0i})\psi_i = \sum_j V_{ij} \psi_j ,$$

where

$$(2.7a) \qquad H_{0i} = \frac{p_0^2}{2\mu_i} + E_i ,$$

$$(2.7b) \qquad V_{ij} = (u_i, V_{0A} u_j) ,$$

where μ_i is the reduced mass in channel i and in our simple example is independent of i. $\mathbf{p}_0$ is the momentum operator of the incident particle. V_{ij} depends only on $\mathbf{r}_0$ but can in general be non-local and energy-dependent.

Eq. (2.6) can be written in matrix notation

$$(2.8) \qquad\qquad (E - H_0)\psi = V\psi ,$$

where the potential V is a square matrix whose elements are given by (2.7b), H_0 is diagonal with elements given by (2.7a), and ψ is a column matrix with elements ψ_i. Hermiticity of V_{0A} implies hermiticity for the matrix V. If the potential V_{0A} is invariant under that particular time reversal transformation which reverses only the motion of the incident particle considered as isolated from the system A, then from invariance under the usually defined time reversal it follows that the matrix V is symmetric.

It is sometimes useful to introduce projection operators P_i for the matrix representation given above:

$$(2.9) \qquad\qquad P_i P_j = \delta_{ij} P_j , \qquad P_i H_0 = P_i H_{0i} = H_0 P_i .$$

If we consider the solutions of the homogeneous equation associated to eq. (2.6):

$$(2.10) \qquad\qquad H_{0i}\varphi_{0i}^{(i)} = \left[\frac{\hbar^2 k_i^2}{2\mu_i} + E_i\right]\varphi_{0i}^{(i)} \equiv E\varphi_{0i}^{(i)} ,$$

where k_i is the wave number in channel i, we can express the projection operators P_i for example in the form

$$(2.11) \qquad\qquad (P_i)_{jk} = \delta_{jk}\delta_{ji} \sum |\varphi_{0i}^{(i)})(\varphi_{0i}^{(i)}| ,$$

where the symbol $\sum$ stands for the integrations and summations over all the continuous and discrete quantum numbers necessary to specify $\varphi_{0i}^{(i)}$ completely. Eq. (2.10) can be written in matrix notation

$$(2.12) \qquad\qquad (E - H_0)\varphi_0^{(i)} = 0 ,$$

where $\varphi_0^{(i)}$ is a column matrix whose elements in the co-ordinate representation are given by

$$(2.13) \qquad\qquad \varphi_{0j}^{(i)}(\boldsymbol{\xi}_i, \boldsymbol{r}) = \delta_{ij}(2\pi)^{-\frac{3}{2}} \chi_{s_i}^{\nu_i} \exp\left[i\boldsymbol{k}_i\cdot\boldsymbol{r}\right] ,$$

where $\boldsymbol{\xi}_i$ indicates the momentum, the spin and its z-component in channel i

$$\boldsymbol{\xi}_i = (\boldsymbol{k}_i, s_i, \nu_i)$$

and $\chi_{s_i}^{\nu_i}$ is the appropriate real spin function in channel i.

All the collision properties are simply expressed in terms of V and ψ; we have in fact for a reaction leading to the channel f

$$(2.14) \qquad (\varphi_{0f}^{(f)} u_f, V_{0A} \Psi) = (\varphi_0^{(f)}, V\psi) \,.$$

Eqs. (2.8), (2.12) and (2.14) show that one can really consider ψ as *the* wave function and $H_0 - V$ as *the* hamiltonian for the many-channel system in substitution of the complicated Ψ and $\mathcal{H}$. In the following we will take over these arguments to the description of the collision of *any* two systems. We will base our investigation of the many-channel problem on the properties of the general set of eq. (8) that we will assume to have been introduced in one way or another for *any* kind of reactions, even for reactions leading to three or more particles (*), in such a way that the collision properties be defined by the right hand side of eq. (2.14). For simplicity the potential V will be taken to be energy-independent.

3. – Time-independent scattering formalism.

In this Section I will consider in some detail by following standard methods ([3,23-25]) the general time-independent scattering formalism which applies to the case of modified Coulomb forces and many channels.

We separate the interaction hamiltonian matrix V into two parts:

$$(3.1) \qquad V = V_c + V' \,,$$

where V_c is the diagonal matrix which describes the Coulomb interaction in the various channels (we assume that Coulomb-like tails exist only in the diagonal elements of V):

$$(3.2) \qquad V_{cj}(r) = Z_{1j} Z_{2j} e^2/r \,.$$

The integral equation for the state vector ψ reads as follows (**)

$$(3.3) \qquad \psi^{(i)} = \varphi_0^{(i)} + G_c V' \psi^{(i)} \,,$$

(*) A generalization of H_0, V and ψ is of course needed in these cases.

([24]) C. Møller: *Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd.*, **23**, No. 1 (1945); B. A. Lippmann and J. Schwinger: *Phys. Rev.*, **79**, 469 (1950); M. Gell-Mann and M. L. Goldberger: *Phys. Rev.*, **91**, 398 (1953).

([25]) M. H. Hull, jr. and G. Breit: *Handbuch der Physik*, Band XLI/1, (Berlin, 1959), p. 408.

(**) Equation (3.3) does not lead to the peculiar amplitude renormalization (see ref. ([25])) one has to perform when the free wave $\varphi_0^{(i)}$ and the free Green's function G_0 are taken in place of the modified $\varphi_c^{(i)}$ and G_c.

where the Coulomb Green's functions G_c is given by

$$(3.4) \qquad G_c = (E + i\varepsilon - H_0 - V_c)^{-1}$$

with the usual $\varepsilon > 0$ device. Like G_c, both $\psi^{(i)}$ and $\varphi_c^{(i)}$ in (3.3) satisfy an outgoing wave boundary condition. Sometime we will place a superscript $(+)$ to these wave functions. The superscript (i) indicates that the incoming wave packet is in the i-th channel. In the co-ordinate representation the Coulomb wave function $\varphi_c^{(i)}$ is given by

$$(3.5) \qquad \varphi_{cj}^{(i)}(\boldsymbol{\xi}_j, \boldsymbol{r}) = \delta_{ij}(2\pi)^{-\frac{3}{2}}\chi_{s_i}^{\nu_i} \sum_l i^l(2l+1)\exp[i\sigma_l^i]P_l\left(\frac{\boldsymbol{k}_i \cdot \boldsymbol{r}}{k_i r}\right)\frac{F_l(k_i r)}{k_i r},$$

where

$$(3.6) \qquad \begin{cases} \sigma_l^i = \arg \Gamma(l+1+i\eta_i) \\ \eta_i = A_i/k_i = Z_{1i}Z_{2i}e^2/\hbar v_i \end{cases}$$

and v_i is the relative velocity of the two particles in channel i. $F_l(\varrho)$ is the real Coulomb wave function regular at the origin in the notation of HULL and BREIT ([25]):

$$(3.7) \qquad F_l(\varrho) = \frac{\exp[-(\pi/2)\eta]}{(2l+1)!}|\Gamma(l+1+i\eta)|2^l\varrho^{l+1}\frac{M_{i\eta,l+\frac{1}{2}}(2i\varrho)}{(2i\varrho)^{l+1}}, \qquad \varrho = kr,$$

with

$$(3.8) \qquad M_{i\eta,l+\frac{1}{2}}(2i\varrho)/(2i\varrho)^{l+1} = 1 + \frac{\eta}{l+1}\varrho + \frac{2\eta^2 - (l+1)}{2(l+1)(2l+3)}\varrho^2 + \dots.$$

Asymptotically we have

$$(3.9) \qquad \lim_{\varrho \to \infty} F_l(\varrho) = \sin\left(\varrho - \eta \log 2\varrho - \frac{l\pi}{2} + \sigma_l\right).$$

It is useful to recall also the definition of the irregular Coulomb wave function $F_l(\varrho) - iG_l(\varrho)$:

$$(3.10) \qquad H_l^{(1)}(\varrho) \equiv F_l(\varrho) - iG_l(\varrho) = \exp\left[\frac{\pi}{2}\eta\right]\exp[i\sigma_l](-i)^{l+1}W_{-i\eta,l+\frac{1}{2}}(-2i\varrho),$$

which asymptotically behaves like (*)

$$(3.11) \qquad \lim_{\varrho \to \infty} H_l^{(1)}(\varrho) = \exp[i(\varrho - \eta \log 2\varrho)](-i)^{l+1}\exp[i\sigma_l]$$

([26]) S. OKUBO and D. FELDMAN: *Phys. Rev.*, **117**, 292 (1960).
(*) $F_l(\varrho)$, $G_l(\varrho)$ and $H_l^{(1)}(\varrho)$ go over to $\varrho j_l(\varrho)$, $-\varrho n_l(\varrho)$ and $\varrho h_l^{(1)}(\varrho)$ for $Z_1 Z_2 e^2 = 0$

and contains therefore only outgoing waves for large ϱ. The functions M and W used above are the regular and irregular confluent hypergeometric functions in the notation of WHITTAKER and WATSON [27].

As can be easily seen from the asymptotic formulas (3.9) and (3.11), the wronskian of F_l with $H_l^{(1)}$ is

$$(3.12) \qquad \frac{\mathrm{d}F_l(\varrho)}{\mathrm{d}\varrho} H_l^{(1)}(\varrho) - F_l(\varrho) \frac{\mathrm{d}H_l^{(1)}(\varrho)}{\mathrm{d}\varrho} = -i \, .$$

Eq. (3.12) together with (3.11), enable us to write for the channel f, in the co-ordinate representation, the Coulomb Green's function G_{cf} satisfying an outgoing wave boundary condition:

$$(3.13) \qquad \langle \boldsymbol{r} \, | \, G_{cf} \, | \, \boldsymbol{r}' \rangle = -\frac{i\mu_f k_f}{2\pi\hbar^2} \sum_l (2l+1) P_l \left(\frac{\boldsymbol{r} \cdot \boldsymbol{r}'}{rr'} \right) \frac{F_l(k_f r_>)}{k_f r_<} \frac{H_l^{(1)}(k_f r_>)}{k_f r_>} \, ,$$

with the usual meaning for $r_<$ and $r_>$.

In terms of the confluent hypergeometric functions M and W, we have

$$(3.14) \qquad \langle \boldsymbol{r} \, | \, G_{cf} \, | \, \boldsymbol{r}' \rangle = -\frac{\mu_f}{2\pi\hbar^2} \sum_l \frac{2^l}{(2l)!} \frac{r_<^l}{r_>} P_l \left(\frac{\boldsymbol{r} \cdot \boldsymbol{r}'}{rr'} \right) \frac{M_{i\eta_f, l+\frac{1}{2}}(2ik_f r_<)}{(2ik_f r_<)^{l+1}} \cdot$$

$$\cdot [(-ik_f)^l \Gamma(l+1+i\eta_f) W_{-i\eta_f, l+\frac{1}{2}}(-2ik_f r_>)] \, .$$

The asymptotic expression for $\langle \boldsymbol{r} \, | \, G_{cf} \, | \, \boldsymbol{r}' \rangle$ is obtained using (3.11):

$$(3.15) \qquad \operatorname*{Lim}_{r \to \infty} \langle \boldsymbol{r} \, | \, G_{cf} \, | \, \boldsymbol{r}' \rangle = -\frac{\mu_f \sqrt{2\pi}}{\hbar^2} \frac{1}{r} \exp \left[i(k_f r - \eta_f \log 2k_f r) \right] \cdot$$

$$\cdot \sum_{s_f \nu_f} \chi_{s_f}^{\nu_f} (\varphi_{cf}^{(f)(-)} (\boldsymbol{\xi}_f) \, | \, \boldsymbol{r}') \, ,$$

where $\boldsymbol{k}_f = k_f \boldsymbol{r}/r$. $\varphi_c^{(f)(-)}$ is the Coulomb wave function which satisfies an incoming wave boundary condition with the outgoing wave packet in channel f:

$$(3.16) \qquad \varphi_{cj}^{(f)(-)}(\boldsymbol{\xi}_f, \boldsymbol{r}) = \delta_{fj}(2\pi)^{-\frac{3}{2}} \chi_{s_f}^{\nu_f} \sum_l i^l (2l+1) \exp \left[-i\sigma_l^f \right] P_l \left(\frac{\boldsymbol{k}_f \cdot \boldsymbol{r}}{k_f r} \right) \frac{F_l(k_f r)}{k_f r} \, ,$$

$\varphi_c^{(f)(-)}$ is related to $\varphi_c^{(f)(+)}$ through time reversal and space reflection:

$$(3.17) \qquad PT \varphi_c^{(f)(-)}(\boldsymbol{k}_f, s_f, \nu_f) = (-)^{s_f + \nu_f} \varphi_c^{(f)(+)}(\boldsymbol{k}_f, s_f, -\nu_f) \, ,$$

[27] E. T. WHITTAKER and G. N. WATSON: *A Course of Modern Analysis* (New York, 1948).

where the operators P and T are defined by (see ref. ([28]))

$$(3.18) \qquad \langle r | P = \langle -r |, \qquad T = \sum_j P_j (i\sigma_y^{(1_j)})(i\sigma_y^{(2_j)}) K ,$$

where K is the complex conjugation operator and $\sigma_y^{(n_j)}$ is the Pauli spin matrix for the n-th particle in channel j if it has spin $\frac{1}{2}$, or $i\sigma_y^{(n_j)} = 1$ if it has spin zero. The sum on j extends over all the channels (*).

The asymptotic behaviour of eq. (3.3) determines the collision properties of the problem. Besides (3.15), we need the asymptotic form of $\varphi_c^{(i)}$ for which we refer for example to ref. ([23]). The result is

$$(3.19) \qquad \lim_{r \to \infty} (2\pi)^{\frac{3}{2}} \chi_{s_f}^{\nu_f *} \psi_f^{(i)} = \exp\left[i(\boldsymbol{k}_i \cdot \boldsymbol{r} + \eta_i \log[k_i r - \boldsymbol{k}_i \cdot \boldsymbol{r}])\right] \delta_{f_i} \delta_{s_f s_i} \delta_{\nu_f \nu_i} +$$

$$+ \frac{1}{r} \exp\left[i(k_f r - \eta_f \log 2k_f r)\right] \left(\frac{v_i}{v_f}\right)^{\frac{1}{2}} \Theta_{fi}(\boldsymbol{\xi}_f, \boldsymbol{\xi}_i) ,$$

where, defining the T-matrix by

$$(3.20) \qquad \Theta_{fi} = -\left(\frac{2\pi}{\hbar}\right)^2 (\mu_i \mu_f k_f / k_i)^{\frac{1}{2}} T_{fi} ,$$

we have

$$(3.21) \quad T_{fi} = T_{ci} \delta_{fi} \delta_{s_f s_i} \delta_{\nu_f \nu_i} + (\varphi_c^{(f)(-)}, V' \psi^{(i)(+)}) \equiv T_{ci} \delta_{fi} \delta_{s_f s_i} \delta_{\nu_f \nu_i} + (\psi^{(f)(-)}, V' \varphi_c^{(i)(+)}) ,$$

T_{ci} describes the scattering in channel i in the absence of everything but Coulomb forces. T_{ci} is related, through (3.20), to the amplitude

$$(3.22) \qquad \Theta_{ci}(\boldsymbol{k}_f, \boldsymbol{k}_i) = -\frac{\eta_i \exp[2i\sigma_0^i]}{2k_i \sin^2 \frac{1}{2}\vartheta} \exp\left[-i\eta_i \log(\sin^2 \frac{1}{2}\vartheta)\right] =$$

$$= \frac{1}{2ik_i} \sum_l (2l+1)(\exp[2i\sigma_l^i] - 1) P_l(\cos\vartheta) , \qquad \vartheta = \widehat{\boldsymbol{k}_f \boldsymbol{k}_i} .$$

The square magnitude of Θ_{fi} is the differential cross-section from channel i to channel f:

$$(3.23) \qquad \frac{d\sigma_{fi}(\boldsymbol{\xi}_f, \boldsymbol{\xi}_i)}{d\Omega_f} = |\Theta_{fi}(\boldsymbol{\xi}_f, \boldsymbol{\xi}_i)|^2 .$$

([28]) E. P. WIGNER: Group Theory, (New York and London, 1959), Ch. 26; G. C. WICK: Ann. Rev. Nucl. Sci., 8, 1 (1958).

(*) For three or more body channels we need as many $(i\sigma_y^{(n_j)})$ quantities as the number of particles in the considered channel j, see ref. ([17]) .

The matrix T given by (3.21) is simply related to the S-matrix defined by

$$(3.24) \qquad S_{fi} = (\psi^{(f)(-)}, \psi^{(i)(+)}) \equiv (\varphi_0^{(f)}, S\varphi_0^{(i)})$$

which shows manifestly the unitarity of the S-operator. $\psi^{(f)(-)}$ is the scattering solution of the complete Schrödinger equation satisfying an incoming wave boundary condition with the outgoing wave packet in channel f; it is related to $\psi^{(f)(+)}$ by an equation of the type (3.17). We have

$$(3.25a) \qquad S_{fi} = S_{cfi} - 2\pi i\, \delta(E - E')(\varphi_c^{(f)(-)}, V'\psi^{(i)}) \,,$$

where

$$(3.25b) \qquad S_{cfi} = (\varphi_c^{(f)(-)}, \varphi_c^{(i)(+)})\, \delta_{fi}\, \delta_{s_f s_i}\, \delta_{v_f v_i}$$

is the S-matrix for pure Coulomb forces.

The unitarity of the S_c-matrix furnishes the equation for Θ_{ci}:

$$(3.26) \qquad \frac{-2\pi i}{k_i}[\Theta_{ci}(\mathbf{k}_f, \mathbf{k}_i) - \Theta_{ci}^*(\mathbf{k}_i, \mathbf{k}_f)] = \int d\Omega'\, \Theta_{ci}^*(\mathbf{k}', \mathbf{k}_f)\Theta_{ci}(\mathbf{k}', \mathbf{k}_i) \,.$$

Use of (3.26) and of the unitarity of the full S-operator yields the following equation for the complete amplitude Θ_{fi}:

$$(3.27) \qquad -2\pi i\left[\frac{1}{k_f}\Theta_{fi}(\boldsymbol{\xi}_f, \boldsymbol{\xi}_i) - \frac{1}{k_i}\Theta_{if}^*(\boldsymbol{\xi}_i, \boldsymbol{\xi}_f)\right] - \sum_{ns_n v_n}\int d\Omega_n\, \Theta_{nf}^*(\boldsymbol{\xi}_n, \boldsymbol{\xi}_f)\Theta_{ni}(\boldsymbol{\xi}_n, \boldsymbol{\xi}_i) \,,$$

where the sum over n runs only over the channels which are open at the considered energy. If in (3.27) we put $i = f$ and $\boldsymbol{\xi}_i = \boldsymbol{\xi}_f$, we get the *optical theorem*:

$$(3.28) \qquad \frac{4\pi}{k_i}\,\mathrm{Im}\,\Theta_{ii}(\boldsymbol{\xi}_i, \boldsymbol{\xi}_i) = \sum_{ns_n v_n}\int d\Omega_n |\Theta_{ni}(\boldsymbol{\xi}_n, \boldsymbol{\xi}_i)|^2 \equiv \sigma_i^{\text{total}}(\boldsymbol{\xi}_i) \,.$$

It is useful sometimes to use the S-matrix expanded in spherical harmonics:

$$(3.29) \qquad S_{fi} = \frac{\hbar^2\delta(E - E')}{(\mu_i \mu_f k_i k_f)^{\frac{1}{2}}}\sum_{JMl_f l_i} Y_{l_f}^{M-v_f}(\mathbf{k}_f)\, Y_{l_i}^{M-v_i*}(\mathbf{k}_i)\, C_{l_i s_i}(J, M;\, M - v_i, v_i)\cdot$$
$$\cdot C_{l_i s_i}(J, M;\, M - v_i, v_i) S_{f_l f s_f, i l_i s_i}^J(E) \,,$$

where $C_{ls}(J, M; m, v)$ and Y_l^m are the Clebsch-Gordan coefficients and the spherical harmonics in the notation and with the phase convention of Blatt

and Weisskopf ([29]). The unitarity of the S-operator implies that the matrix $S^{J}_{f\,l_{f}s_{f},\,i\,l_{i}s_{i}}$ is *unitary*.

Use of (3.21), (3.22) and (3.25) yields the expansion of Θ_{fi} in terms of the S-matrix, in which the pure Coulomb scattering is separated out:

$$(3.30) \qquad \Theta_{fi}(\boldsymbol{\xi}_{f}, \boldsymbol{\xi}_{i}) = \Theta_{ci}(\boldsymbol{k}_{f}, \boldsymbol{k}_{i})\delta_{fi}\delta_{s_{f}s_{i}}\delta_{\nu_{f}\nu_{i}} +$$

$$+ \frac{2\pi i}{k_{i}} \sum_{JMl_{f}l_{i}} Y^{M-\nu_{f}}_{l_{f}}(\boldsymbol{k}_{f})\, Y^{M-\nu_{i}*}_{l_{i}}(\boldsymbol{k}_{i})\, C_{l_{f}s_{f}}(J,\, M;\, M - \nu_{f},\, \nu_{f}) \cdot$$

$$\cdot C_{l_{i}s_{i}}(J,\, M;\, M - \nu_{i},\, \nu_{i})\, (\exp\,[2i\sigma^{i}_{l_{i}}]\delta_{fi}\delta_{l_{f}l_{i}}\delta_{s_{f}s_{i}} - S^{J}_{fl_{f}s_{f},il_{i}s_{i}}) \cdot$$

Using (3.30) and (3.21) we get the S-matrix in terms of the potential V':

$$(3.31) \qquad S^{J}_{fl_{f}s_{f},\,il_{i}s_{i}} = \exp\,[2i\sigma^{i}_{l_{i}}]\delta_{fi}\delta_{l_{f}l_{i}}\delta_{s_{f}s_{i}} - \frac{i}{\pi\hbar^{2}}\left(\frac{\mu_{i}\mu_{f}}{k_{i}k_{f}}\right)^{\frac{1}{2}} i^{l_{i}-l_{f}}\exp\,[i\sigma^{f}_{l_{f}}] \cdot$$

$$\cdot \sum_{jl_{j}s_{j}} \int dr'\, dr''r'r''\, F_{l_{f}}(k_{f}r')\, V'^{J}_{fl_{f}s_{f},jl_{j}s_{j}}(r'\,|\,r'')\, \psi^{J}_{jl_{j}s_{j}}(k_{i}l_{i}s_{i},\, r'') ,$$

where we have supposed V' to be in general non-local (*). $V'^{J}(r'\,|\,r'')$ and $\psi^{J}(r)$ are defined by

$$(3.32) \qquad \int d\Omega'\, d\Omega''\, \mathscr{Y}^{M'*}_{J'l_{f}s_{f}}(\boldsymbol{r}')\langle \boldsymbol{r}'\,|\,V'_{fj}\,|\,\boldsymbol{r}''\rangle\, \mathscr{Y}^{M}_{Jl_{j}s_{j}}(\boldsymbol{r}'') = V'^{J}_{fl_{f}s_{f},jl_{j}s_{j}}(r'\,|\,r'')\delta_{JJ'}\delta_{MM'} ,$$

$$(3.33) \qquad \psi^{(i)(+)}_{j}(\boldsymbol{\xi}_{i},\, \boldsymbol{r}) = (2\pi)^{-\frac{3}{2}} \sum_{JMl_{j}s_{j}l_{i}} C_{l_{i}s_{i}}(J,\, M;\, M - \nu_{i},\, \nu_{i})\, Y^{M-\nu_{i}*}_{l_{i}}(\boldsymbol{k}_{i}) \cdot$$

$$\cdot \mathscr{Y}^{M}_{Jl_{j}s_{j}}(\boldsymbol{r})\, i^{l_{i}}\, \frac{\psi^{J}_{jl_{j}s_{j}}(k_{i}l_{i}s_{i},\, r)}{k_{i}\,r} ,$$

where

$$(3.34) \qquad \mathscr{Y}^{M}_{Jls}(\boldsymbol{r}) = \sum_{m\nu} C_{ls}(J,\, M;\, m,\, \nu)\, Y^{m}_{l}(\boldsymbol{r})\chi^{\nu}_{s}$$

is a total angular momentum eigenfunction.

$\psi^{J}(r)$ satisfies the differential equation:

$$(3.35) \qquad \left[k^{2}_{j} + \frac{d^{2}}{dr^{2}} - \frac{l_{j}(l_{j}+1)}{r^{2}} - \frac{2\mu_{j}}{\hbar^{2}}\, Z_{1j}Z_{2j}\, \frac{e^{2}}{r}\right] \psi^{J}_{jl_{j}s_{j}}(k_{i}l_{i}s_{i},\, r) =$$

$$= \sum_{kl_{k}s_{k}} \frac{2\mu_{j}}{\hbar^{2}} \int_{0}^{\infty} dr'\, rr'\, V'^{J}_{jl_{j}s_{j},kl_{k}s_{k}}(r\,|\,r')\, \psi^{J}_{kl_{k}s_{k}}(k_{i}l_{i}s_{i},\, r') ,$$

([29]) J. M. Blatt and V. F. Weisskopf: *Theoretical Nuclear Physics* (New York, 1952), Appendix A.

(*) In the local limit $V'^{J}(r'\,|\,r'') \to (\delta(r'-r'')/r'^{2})V'^{J}(r')$.

or, in turn, the integral equation:

$$(3.36) \qquad \psi^{J}_{jl_js_j}(k_il_is_i, r) = 4\pi \exp\left[i\sigma^i_{l_i}\right]F_{l_i}(k_ir)\delta_{ji}\delta_{l_jl_i}\delta_{s_js_i} -$$

$$-\frac{i}{k_j}\int dr'\, dr''r'r''\, F_{l_j}(k_jr_<)H^{(1)}_{l_j}(k_jr_<) \sum_{kl_ks_k}\frac{2\mu_j}{\hbar^2}V^{'J}_{jl_js_j,\,kl_ks_k}(r'\ r'')\psi^{J}_{kl_ks_k}(k_il_is_i, r''),$$

where $r_<$ and $r_>$ refer to r and r'. The asymptotic form of (3.36) is related, through (3.31), to the S-matrix:

$$(3.37) \quad \lim_{r\to\infty}\psi^{J}_{jl_js_j}(k_il_is_i, r) = 2\pi i^{l_i+1}\left[\delta_{ji}\delta_{l_jl_i}\delta_{s_js_i}\exp\left[-i(k_ir - \eta_i\log 2k_ir)\right] - \right.$$

$$\left. -(-)^{l_i}\left(\frac{k_i\mu_j}{k_j\mu_i}\right)^{\frac{1}{2}}\exp\left[i(k_jr - \eta_j\log 2k_jr)\right]S^{J}_{jl_js_j,\,il_is_i}\right].$$

Use of time-reversal invariance and of (3.17) furnishes the *principle of detailed balance*:

$$(3.38) \qquad \Theta_{fi}(\boldsymbol{\xi}_f, \boldsymbol{\xi}_i) = (-)^{s_f+\nu_f+s_i+\nu_i}\Theta_{if}(-\boldsymbol{\xi}_i, -\boldsymbol{\xi}_f)\frac{k_f}{k_i},$$

where by $-\boldsymbol{\xi}$ we mean

$$-\boldsymbol{\xi} = (-\boldsymbol{k}, s, -\nu).$$

Comparison of (3.30) and (3.38) shows that, under the hypothesis of time-reversal invariance, the matrix $S^{J}_{fl_fs_f,\,il_is_i}$, is *symmetric* (*) (see also ref. [17] for three-body channels).

We now proceed to the study of the properties of the cross-sections for scattering and reactions just above the threshold of the initial or final channel.

4. – Low-energy behaviour of cross-sections.

We will consider here the behaviour of the various elements of the transition amplitude when either the final channel f is considered at its own threshold (endoergic case) or the initial channel i is considered in the limit $k_i = 0$ (exoergic case). We will consider first the cases $i \neq f$, after that the properties for the scattering case will be deduced as a consequence. We will consider, quite in general, the possibility that Coulomb forces are present in the various

(*) We remind of the rules:

$$Y^{m*}_l = (-)^m Y^{-m}_l, \qquad C_{1s}(J, -M; -m, -\nu) = (-)^{l+s-J}C_{1s}(J, M; m, \nu).$$

channels; the case of no Coulomb interactions will be obtained by simply placing $e^2 Z_1 Z_2 = 0$ in the final formulae.

The following considerations will hold provided the potential V' vanishes beyond a finite value of r. In most cases of interest this condition can be relaxed though. Let us first consider the

a) *Endoergic case.* For an endoergic reaction channel $i \to$ channel f (two-body), an expression for the transition amplitude Θ_{fi} in terms of the potential V' is obtained using (3.16) and the first version of (3.21). If we take the incident beam in the direction of the z-axis, we get

$$(4.1) \quad \begin{cases} \Theta_{fi}(\xi_f, \xi_i) = \sum_{l_f} \Theta_{fi}^{(l_f)}(\xi_f, \xi_i)\, Y_{l_f}^{\nu_i - \nu_f}(\mathbf{k}_f)\,, \\[2mm] \Theta_{fi}^{(l_f)}(\xi_f, \xi_i) = -\dfrac{1}{2\pi^{\frac12}}\left(\dfrac{\mu_i}{\mu_f k_i^3 k_f}\right)^{\frac12} \sum_{Jl_i} (2l_i + 1)^{\frac12} i^{l_i - l_f} \exp\left[i\sigma_{l_f}^f\right] \cdot \\[2mm] \quad \cdot C_{l_f s_f}(J, \nu_i;\, \nu_i - \nu_f, \nu_f)\, C_{l_i s_i}(J, \nu_i;\, 0, \nu_i) \cdot \\[2mm] \quad \cdot \displaystyle\int \mathrm{d}r'\, \mathrm{d}r''\, r' r''\, F_{l_f}(k_f r') \sum_{j l_j s_j} \dfrac{2\mu_f}{\hbar^2}\, V_{f l_f s_f,\, j l_j s_j}^{l'J}(r'\,|\,r'')\, \psi_{j l_j s_j}^J(k_i l_i s_i, r'')\,. \end{cases}$$

In the limit as k_f goes to zero, one can determine from (3.7) the low energy behaviour of the function F_l. Since $M_{i\eta,\,l+\frac12}(2i\varrho)/(2i\varrho)^{l-1}$ goes to a constant, the behaviour of $\Theta_{fi}^{(l_f)}$ at $k_f = 0$ is determined by the function (*)

$$k_f^{l_f + \frac12} \exp\left[-i\sigma_0^f\right] \exp\left[-\eta_f \pi/2\right] \Gamma(l_f + 1 + i\eta_f)/l_f!\,.$$

The limit gives

$$(4.2) \quad \lim_{k_f \to 0} \Theta_{fi}^{(l_f)} \sim \begin{cases} k_f^{l_f + \frac12} & A_f = 0\,, \\[2mm] \sqrt{2\pi} \exp\left[-\eta_f \pi\right] A_f^{l_f + \frac12} i^{l_f}/l_f! & A_f > 0\,, \\[2mm] \sqrt{2\pi}\,|A_f|^{l_f + \frac12}(-i)^{l_f}/l_f! & A_f < 0\,, \end{cases}$$

(endoergic case, $f \neq i$).

We see that in the case of Coulomb repulsion $(A_f > 0)$ in the final channel f, $\Theta_{fi}^{(l_f)}$ starts from $k_f = 0$ like the exponential $\exp\left[-\pi A_f/k_f\right]$, while for Coulomb attraction $(A_f < 0)$ for all l_f, $\Theta_{fi}^{(l_f)}$ is finite there. From the low energy properties of (3.8), we see that the next term in the expansion of $\Theta_{fi}^{(l_f)}$ for $A_f < 0$ is proportional to k_f^2. We have therefore that the derivative of the reaction cross-section σ_{fi} with respect to k_i is finite at threshold $(k_f = 0)$. The case of no Coulomb forces yields the well known energy dependence. We now proceed to the discussion of the

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(*) The term $\exp\left[-i\sigma_0^f\right]$ has been introduced to avoid unobservable oscillations at zero energy.

b) *Exoergic case.* In order to obtain the behaviour of the transition amplitude Θ_{fi} when the initial channel i is considered at its own threshold $(k_i = 0)$, it is convenient to take into consideration the second version of (3.21). Following the same procedure as before, in the limit as k_i goes to zero, we get (*).

$$(4.3) \qquad \underset{k_i \to 0}{\text{Lim}}\ \Theta_{fi}^{(l_i)} \sim \begin{cases} k_i^{l_i - \frac{1}{2}} & A_i = 0, \\ \sqrt{2\pi}\, k_i^{-1} \exp\left[-\pi \eta_i\right] A_i^{l_i + \frac{1}{2}} i^{l_i}/l_i\,! & A_i > 0, \\ \sqrt{2\pi}\, k_i^{-1} |A_i|^{l_i + \frac{1}{2}} (-i)^{l_i}/l_i\,! & A_i < 0, \end{cases}$$

(exoergic case, $f \neq i$).

The characteristic exponential $\exp\left[-\pi A_{ii}/k_i\right]$ is still present for Coulomb repulsion. In the case of Coulomb attraction, $\Theta_{fi}^{(l_i)}$ goes to infinity in the limit $k_i \to 0$. The energy dependence for $A_i = 0$ is well known and yields the $1/v$ law for the absorption of slow neutrons by nuclei [1,3].

c) *Elastic scattering.* If Coulomb forces are present in channel i, then the combination of both (4.2) and (4.3) gives us the energy dependence of the non-Coulomb part of the scattered wave. Consideration of the pure Coulomb scattering amplitude Θ_{ci} shows that at very low energy only pure Coulomb scattering survives, since $\underset{k_i \to 0}{\text{Lim}}\ \Theta_{ci} \sim k_i^{-2}$.

In the case of no Coulomb forces, writing $\Theta_{ii}(\boldsymbol{\xi}'_i, \boldsymbol{\xi}_i) = \sum_{l_i l_i'} \Theta_{ii}^{(l_i', l_i)}(\boldsymbol{\xi}'_i, \boldsymbol{\xi}_i) Y_{l_i}^{\nu_i - \nu_i'}(\boldsymbol{k}'_i)$ we get the well-known formula

$$(4.4) \qquad \underset{k_i \to 0}{\text{Lim}}\ \Theta_{ii}^{(l_i, l_i)} \sim k_i^{l_i} k_i^{l_i}, \qquad A_i = 0 \qquad \text{(scattering, } f = i\text{)},$$

which shows that the scattering cross-section is finite at zero energy when Coulomb forces are not present.

Use of eq. (4.2) and (4.3), in conjunction with (3.30), gives us the threshold behaviour of the various elements of the S-matrix:

$$(4.5) \qquad \exp\left[2i\sigma_{l_i}^i\right] \delta_{fi}\, \delta_{l_f l_i}\, \delta_{s_f s_i} - S_{f l_f s_f,\, i l_i s_i}^J = K_{f l_f s_f}\, \mathscr{M}_{f l_f s_f,\, i l_i s_i}^J K_{i l_i s_i},$$

where

$$(4.6) \qquad K_{f l_f s_f} = \begin{cases} k_f^{l_f + \frac{1}{2}} & A_f = 0, \\ \sqrt{2\pi} \exp\left[-\eta_f \pi\right] A_f^{l_f + \frac{1}{2}} i^{l_f}/l_f\,! & A_f > 0, \\ \sqrt{2\pi} |A_f|^{l_f + \frac{1}{2}} (-i)^{l_f}/l_f\,! & A_f < 0, \end{cases}$$

with a similar expression for $K_{i l_i s_i}$. $\mathscr{M}_{f l_f s_f,\, i l_i s_i}^J$ is a quantity which is finite

(*) Eq. (4.3) can be clearly obtained from (4.2) using the principle of detailed balance (3.38).

and in general non-zero as either k_f or k_i approaches zero. The next term in the low energy expansions of $\mathscr{M}_{fi}^J$ for $A_f < 0$ (for $A_i < 0$) is proportional to k_f^2 (to k_i^2).

If there are no Coulomb forces in channel a and if we consider energies near the threshold of that channel, then it can be easily seen by the use of (4.5) and (3.30) that

(4.7)
$$\begin{cases} \Theta_{ai}(\boldsymbol{\xi}_a, \boldsymbol{\xi}_i) \simeq \delta_{\nu_a \nu_i} \Theta_{ai}^{(l_a=0)} \dfrac{1}{\sqrt{4\pi}}, \\[2mm] \Theta_{an}(\boldsymbol{\xi}_a', \boldsymbol{\xi}_a) \simeq \delta_{\nu_a' \nu_a} \delta_{s_a' s_a} \Theta_{aa}^{(l_a'=0,\, l_a=0)} \dfrac{1}{\sqrt{4\pi}}. \end{cases}$$
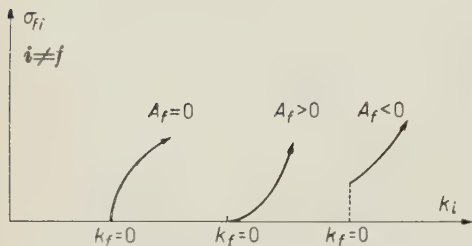
Fig. 1. – Energy behaviour of σ_{fi} vs. the momentum k_i, in the endoergic case.

In Fig. 1 the energy behaviour of the cross-sections (differential or integrated over angles) is plotted for the endoergic case ($f \neq i$) vs. the momentum k_i of the initial channel i. We notice that in this case σ_{fi} starts with an infinite slope in the case of no Coulomb forces in the final channel f. If we define $\sigma_{fi} = \sum_{l_f} \sigma_{fi}^{(l_f)} \equiv$
$= \sum_{l_f} |\Theta_{fi}^{(l_f)}|^2$ for the reaction cross-section integrated over angles (*), remembering that $dk_f/dk_i = \mu_f k_i / \mu_i k_f$, for the partial cross-section $\sigma_{fi}^{(l_f)}$ we get

(4.8)
$$\operatorname*{Lim}_{k_f \to 0} \frac{d\sigma_{fi}^{(l_f)}}{dk_i} \sim (2l_f + 1)\, k_f^{2l_f - 1} k_i \begin{cases} = \infty, & l_f = 0, \\ = 0, & l_f > 0, \end{cases}$$

(endoergic case, $f \neq i$, $A_f = 0$).

So that we recognize that the $l_f = 0$ part of σ_{fi} exhibits the infinite derivative with respect to k_i at $k_f = 0$. Notice that the derivative of σ_{fi} (differential or integrated over angles, indifferently) with respect to k_f, is finite at $k_f = 0$.

If there are Coulomb forces in channel f, then the reaction cross-section σ_{fi} (differential or integrated over angles, endoergic case) starts with zero slope in the repulsive case, with finite slope and non-zero value in the attractive case.

(*) The same is true for the differential cross section; in that case we define

$$\sigma_{fi}^{(0)} = |\Theta^{(0)}|^2 \frac{1}{4\pi}, \qquad \sigma_{fi}^{(1)} = 2\,\mathrm{Re}\left\{\Theta_{fi}^{(0)} \Theta_{fi}^{(1)*} \frac{1}{\sqrt{4\pi}} \; Y_1^{\nu_i - \nu_f *}(\boldsymbol{k}_f)\right\},$$

$$\sigma_{fi}^{(2)} = 2\,\mathrm{Re}\left\{\Theta_{fi}^{(0)} \Theta_{fi}^{(2)*} \frac{1}{\sqrt{4\pi}} \; Y_2^{\nu_i - \nu_f *}(\boldsymbol{k}_f)\right\} + |\Theta_{fi}^{(1)} Y_1^{\nu_i - \nu_f}(\boldsymbol{k}_f)|^2, \quad \text{etc.}$$

d) Three-body final channel. We want to consider here the low-energy properties of an endoergic reaction cross-section leading to a final three-body channel. Let us consider first the case with no Coulomb forces in channel f. Formulae (3.19), (3.20) and (3.21) are easily generalized to cover this case. In particular we have for the transition amplitude Θ_{fi}

$$(4.9) \qquad \Theta_{fi}(\boldsymbol{\xi}_1, \boldsymbol{\xi}_{23}; \boldsymbol{\xi}_i) = -\left(\frac{2\pi}{\hbar}\right)^2 (\mu_i \mu_{23} k_{23}/k_i)^{\frac{1}{2}} (\varphi_0^{(f)}(\boldsymbol{\xi}_1)\, \varphi_0^{(f)}(\boldsymbol{\xi}_{23}), V\psi^{(i)}) .$$

Θ_{fi}, as given here, yields the cross-section for the reaction leading to the three-body channel f in which the first particle (n. 1) obtains a momentum between $\hbar\boldsymbol{k}_1$ and $\hbar\boldsymbol{k}_1 + \mathrm{d}\hbar\boldsymbol{k}_1$, and the particles 2 and 3 receive a relative momentum $\hbar k_{23}$ in the direction between Ω_{23} and $\Omega_{23} + \mathrm{d}\Omega_{23}$:

$$(4.10) \qquad \bar{\sigma}_{fi} \equiv \frac{\mathrm{d}\sigma_{fi}}{\mathrm{d}^3 k_1\, \mathrm{d}\Omega_{23}} = |\Theta_{fi}(\boldsymbol{\xi}_1, \boldsymbol{\xi}_{23}; \boldsymbol{\xi}_i)|^2 ,$$

$\varphi_0^{(f)}(\boldsymbol{\xi}_1)$ and $\varphi_0^{(f)}(\boldsymbol{\xi}_{23})$ are given as follows:

$$(4.11) \qquad \begin{cases} \varphi_0^{(f)}(\boldsymbol{\xi}_1, \boldsymbol{r}_1) &= (2\pi)^{-\frac{3}{2}}\chi_1 \exp\left[i\boldsymbol{k}_1 \cdot \boldsymbol{r}_1\right] , \\ \varphi_0^{(f)}(\boldsymbol{\xi}_{23}, \boldsymbol{r}_{23}) &= (2\pi)^{-\frac{3}{2}}\chi_{23} \exp\left[i\boldsymbol{k}_{23} \cdot \boldsymbol{r}_{23}\right] , \end{cases}$$

where $\boldsymbol{r}_1$ is the distance separating particle 1 from the center of mass of the system $2+3$; $\boldsymbol{r}_{23}$ is the relative distance between 2 and 3. μ_i is the reduced mass in the incoming two-body channel i, μ_{23} is the reduced mass of the system $2+3$ in the final channel f.

We assume, as before, that the interaction hamiltonian V vanishes beyond finite values of r_1 and r_{23}. In the limit as the total energy E approaches the threshold for the channel f from above, we have

$$(4.12) \qquad \operatorname*{Lim}_{E \to E_f} \Theta_{fi}(l_1, l_{23}) \simeq k_1^{l_1} k_{23}^{l_{23}+\frac{1}{2}} ,$$

where we have again separated Θ_{fi} into the contributions from the various values of l_1 and l_{23}, orbital angular momentum of particle 1 and of particle 2 relative to 3, respectively. It is clear that (4.12) holds also in the limit as either k_{23} or k_1 approaches zero, the total energy E being kept fixed in the limiting process.

The following property of $\bar{\sigma}_{fi}$ is established:

$$(4.13) \qquad \operatorname*{Lim}_{k_{23} \to 0} \frac{\mathrm{d}\bar{\sigma}_{fi}}{\mathrm{d}k_{23}} = \text{finite} ,$$

$\bar{\sigma}_{fi}$ is zero for $k_{23} = 0$, for $k_1 = 0$ and also for $E = E_f$, *i.e.* right at the threshold for channel f (*). Remembering that

$$\frac{\hbar^2 k_{23}^2}{2\mu_{23}} - \frac{\hbar^2 k_i^2}{2\mu_i} - \frac{\hbar^2 k_1^2}{2\mu_1} - E_i - E_f ,$$

where

$$\mu_1 = \frac{m_1(m_2 + m_3)}{m_1 + m_2 + m_3} ,$$

we get for the integrated cross-section:

(4.14)
$$\left\{ \begin{array}{l}
\displaystyle \mathop{\mathrm{Lim}}_{E \to E_f} \int_0^{k_1^{\max}} \mathrm{d}^3 k_1 \, \bar{\sigma}_{fi} = 0 , \\[2em]
\displaystyle \mathop{\mathrm{Lim}}_{E \to E_f} \frac{\mathrm{d}}{\mathrm{d}k_i} \int_0^{k_1^{\max}} \mathrm{d}^3 k_1 \, \bar{\sigma}_{fi} = 0 , \\[2em]
\displaystyle \mathop{\mathrm{Lim}}_{E \to E_f} \frac{\mathrm{d}^2}{\mathrm{d}k_i^2} \int_0^{k_1^{\max}} \mathrm{d}^3 k_1 \, \bar{\sigma}_{fi} = \text{finite} .
\end{array} \right.$$

The second derivative of the integrated cross-section with respect to k_i, is in general different from zero, and finite.

If two of the particles present in channel f are charged, labelling them with the number 2 and 3, we find for $\Theta_{fi}(l_1, l_{23})$, as a function of k_{23} in the limit $k_{23} \to 0$, the behaviour (4.2). For Coulomb repulsion we have that also the second derivative of the integrated cross-section with respect to k_i is zero in the limit $E \to E_f$ (**). For Coulomb attraction, while the integrated cross-section and its first derivative are still zero, the second derivative turns out to be infinite.

5. – Threshold effects.

We are interested now in the study of the energy behaviour of the various elements of the transition amplitude at the threshold for the channel a, a new

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(*) Notice that for $\bar{\sigma}_{fi}$ only the part relative to $l_{23} = 0$ contributes to (4.13), waves with $l_{23} > 0$ give zero contribution in the limit as k_{23} goes to zero.

(**) This is true also in the case in which the third particle has a charge of the same sign of the other two.

channel which is opening up at the considered energy. In this section we will consider the case in which Coulomb forces are not present in channel a. Coulomb effects will be discussed in Section **6**. The discussion will be led for $a \neq i$ and $a \neq f$, since the other cases have been considered in Section **4**.

Our method starts with the realization that the complete Green's function $\mathfrak{G}$, defined by the equation

$$(5.1) \qquad (E - H_0 - V)\mathfrak{G} = 1$$

and the outgoing wave boundary condition, must contain all the information we want to get on the energy behaviour of the various cross-sections at the threshold for channel a. In fact $\mathfrak{G}$ appears in the transition matrix as follows:

$$(5.2) \qquad T_{fi} = T_{ci}\delta_{fi}\delta_{s_f s_i}\delta_{\nu_f \nu_i} + \left(\varphi_c^{(f)(-)}, [V' + V'\mathfrak{G}V']\varphi_c^{(i)(+)}\right) ;$$

since $a \neq i$, $a \neq f$ and T_{ci} has nothing to do with the channel a, all the threshold effects must be in $\mathfrak{G}$.

To work this out we have to isolate the part in $\mathfrak{G}$ which refers to channel a only. This is most easily done by the introduction of two projection operators, P_a and P_b, the P_a refers to the new channel a and has been defined in Section 2, and P_b is given by $P_b = \sum_{j \neq a} P_j$, where the sum extends over all the other channels. This separation has been carried out in all details on p. 498 of ref. [30], and we give here the result:

$$(5.3) \qquad \mathfrak{G} = G_b + (1 + G_b V)P_a \mathscr{G}_a P_a(VG_b + 1) ,$$

where G_b and $\mathscr{G}_a$ are Green's functions which satisfy the equations:

$$(5.4) \qquad (E - H_0 P_b - P_b V P_b)G_b = P_b ,$$

$$(5.5) \qquad [E - \mathscr{H}_a(E)]\,\mathscr{G}_a = 1 ,$$

with the outgoing wave boundary condition. $\mathscr{H}_a(E)$ is the effective hamiltonian for channel a:

$$(5.6) \qquad \mathscr{H}_a(E) = H_{0a} + V_{aa} + \sum_{b'b'' \neq a} V_{ab'}[G_b(E)]_{b'b''}V_{b''a} ,$$

which is clearly energy-dependent and, since at the considered energy some channels b' are open, non-hermitian. In the following the term with the sum, at the right hand side of (5.6), will be indicated shortly as $V_{ab}G_b V_{ba}$.

[30] L. FONDA and R. G. NEWTON: *Ann. Phys.*, **10**, 490 (1960).

The various elements of G_b depend only on the momenta in channels b'. Therefore when we take the derivative of G_b with respect to k_a for $k_a = 0$ (threshold of channel a) we get zero. Near the threshold we have then

$$(5.7) \qquad \frac{\partial \mathcal{G}}{\partial |k_a|} = (1 + G_b V) P_a \frac{\partial \mathcal{G}_a}{\partial |k_a|} P_a (V G_b + 1) ,$$

where we have taken the derivative with respect to the modulus of k_a so that eq. (5.7) holds both above and below the threshold for the channel a. We remind that k_a goes into $i|k_a|$ below the threshold (see ref. ([1,5,8])). This can be most easily understood from eq. (5.10) below (*).

It is by now clear that all the threshold effects are in $\mathcal{G}_a$ and they can be best exploited by considering the integral equation for $\mathcal{G}_a$:

$$(5.8) \qquad \mathcal{G}_a = G_{0a} + G_{0a}(V_{aa} + V_{ab} G_b V_{ba}) \mathcal{G}_a ,$$

where G_{0a} satisfies the equation

$$(5.9) \qquad (E - H_{0a}) G_{0a} = 1$$

and the outgoing wave boundary condition. In the co-ordinate representation:

$$(5.10) \qquad \langle r | G_{0a} | r' \rangle = -\frac{\mu_a}{2\pi\hbar^2} \frac{1}{|r - r'|} \cdot \begin{cases} \exp[ik_a|r - r'|] & \text{above ,} \\ \exp[-|k_a||r - r'|] & \text{below .} \end{cases}$$

We take now the derivative with respect to $|k_a|$ of (5.8) near the threshold:

$$(5.11) \qquad \frac{\partial \mathcal{G}_a}{\partial |k_a|} = \frac{\partial G_{0a}}{\partial |k_a|} \{1 + (V_{aa} + V_{ab} G_b V_{ba}) \mathcal{G}_a\} + G_{0a}(V_{aa} + V_{ab} G_b V_{ba}) \frac{\partial \mathcal{G}_a}{\partial |k_a|} .$$

Eq. (5.11), together with (5.10), establishes the important fact that $\partial \mathcal{G}_a / \partial |k_a|$ contains only outgoing waves. If we multiply (5.11) on the left by G_{0a}^{-1} we get the equation

$$(5.12) \qquad [E - \mathcal{H}_a(E)] \frac{\partial \mathcal{G}_a}{\partial |k_a|} = G_{0a}^{-1} \frac{\partial G_{0a}}{\partial |k_a|} \{1 + (V_{aa} + V_{ab} G_b V_{ba}) \mathcal{G}_a\} .$$

(*) The rule holds true also when Coulomb forces are present in channel a.

(5.12) can be solved formally:

$$(5.13) \qquad \frac{\partial \mathscr{G}_a}{\partial |k_a|} = \mathscr{G}_a G_{0a}^{-1} \frac{\partial G_{0a}}{\partial |k_a|} \{1 + (V_{aa} + V_{ab}G_b V_{ba})\mathscr{G}_a)\} \equiv$$

$$\equiv \{1 + \mathscr{G}_a(V_{aa} + V_{ab}G_b V_{ba})\} \frac{\partial G_{0a}}{\partial |k_a|} \{1 + (V_{aa} + V_{ab}G_b V_{ba})\mathscr{G}_a\} .$$

Since $\partial \mathscr{G}_a / \partial |k_a|$ contains only outgoing waves, no solutions of the homogeneous equation $(E - \mathscr{H}_a(E))u = 0$ can be added to (5.13). In fact the non-hermitian hamiltonian $\mathscr{H}_a(E)$ contains a skew-hermitian term $- i\pi V_{ab} \cdot$ $\cdot \delta(E - H_0 P_b - P_b V P_b) V_{ba}$ which prevents the presence of only outgoing waves in u.

We are left with the discussion of the derivative of G_{0a}. Writing G_{0a} in the form

$$(5.14) \qquad \begin{cases} \langle \boldsymbol{r} | G_{0a} | \boldsymbol{r}' \rangle = \sum_{l_a} (2l_a + 1) P_{l_a}\left(\frac{\boldsymbol{r} \cdot \boldsymbol{r}'}{rr'}\right) G_{0a}^{(l_a)}(r|r') , \\[2mm] G_{0a}^{(l_a)}(r|r') = - \frac{i\mu_a}{2\pi\hbar^2} k_a j_{l_a}(k_a r_<) h_{l_a}^{(1)}(k_a r_>) . \end{cases}$$

we obtain near the threshold:

$$(5.15) \qquad \frac{\partial G_{0a}^{(l_a)}}{\partial |k_a|} = - \frac{i\mu_a}{2\pi\hbar^2} \binom{1}{i} \delta_{l_a,0} .$$

(5.15) tells us that the threshold effects come out only from the S-wave of channel a. $\binom{a}{b}$ is a symbol to indicate that above the threshold the upper quantity has to be used, below threshold the lower.

Substitution of (5.15) into (5.13) and of the result into (5.7), after using the identity at the threshold

$$(5.16) \qquad (2\pi)^{-3} P_a \int \mathrm{d}^3 r \, \mathrm{d}^3 r' \, | \boldsymbol{r} \rangle \langle \boldsymbol{r}' | P_a = \sum_{s_a \nu_a} | \varphi_0^{(a)} \rangle \langle \varphi_0^{(a)} | , \qquad \text{for } k_a = 0 ,$$

gives the final equation for the transition amplitude at threshold [6]:

$$(5.17) \qquad \frac{\partial}{\partial |k_a|} \Theta_{fi}(\boldsymbol{\xi}_f, \boldsymbol{\xi}_i) = \binom{i}{-1} \sum_{s_a \nu_a} \Theta_{fa}(\boldsymbol{\xi}_f, \boldsymbol{\xi}_a) \Theta_{ai}(\boldsymbol{\xi}_a, \boldsymbol{\xi}_i) \delta_{\nu_a \nu_i} .$$

Only the S-wave of channel a contributes to (5.17). The Kronecker's $\delta_{\nu_a \nu_i}$ comes from (4.7).

The derivative of Θ_{fi} with respect to any other channel momentum is *infinite* both from above and from below, since

$$(5.18) \qquad \lim_{k_a \to 0} \frac{\mathrm{d}|k_a|}{\mathrm{d}k_{b'}} = \binom{+}{-} \frac{\mu_a}{\mu_{b'}} \frac{k_{b'}}{|k_a|} = \binom{+}{-} \infty .$$

From (5.17) one can easily obtain the derivative of the differential cross-section

$$(5.19) \qquad \frac{\partial}{\partial |k_a|} \frac{d\sigma_{fi}(\boldsymbol{\xi}_f, \boldsymbol{\xi}_i)}{d\Omega_f} = -2 \binom{\mathrm{Im}}{\mathrm{Re}} \sum_{s_a \nu_a} \Theta^*_{fi}(\boldsymbol{\xi}_f, \boldsymbol{\xi}_i)\, \Theta_{fa}(\boldsymbol{\xi}_f, \boldsymbol{\xi}_a)\, \Theta_{ai}(\boldsymbol{\xi}_a, \boldsymbol{\xi}_i)\, \delta_{\nu_a \nu_i} ,$$

and of the total cross-section defined by (3.28)

$$(5.20) \qquad \frac{\partial}{\partial |k_a|} \sigma^{\mathrm{tot}}_i(\boldsymbol{\xi}_i) = \frac{4\pi}{k_i} \binom{\mathrm{Re}}{-\mathrm{Im}} \sum_{s_a \nu_a} \Theta_{ia}(\boldsymbol{\xi}_i, \boldsymbol{\xi}_a)\, \Theta_{ai}(\boldsymbol{\xi}_a, \boldsymbol{\xi}_i)\delta_{\nu_a \nu_i} .$$

Eq. (5.17) holds true also for $\Theta^{(l_f)}_{fi}(\xi_f, \xi_i)$ with the substitution of $\Theta_{fa}(\boldsymbol{\xi}_f, \boldsymbol{\xi}_a)$ with $\Theta^{(l_f)}_{fa}(\xi_f, \xi_a)$ on the right-hand side. The following equation for the l_f-partial cross-section integrated over angles is then obtained:

$$(5.21) \qquad \frac{\partial}{\partial |k_a|} \sigma^{(l_f)}_{fi} = -2 \binom{\mathrm{Im}}{\mathrm{Re}} \sum_{s_a \nu_a} \Theta^{(l_f)*}_{fi}(\xi_f, \xi_i)\, \Theta^{(l_f)}_{fa}(\xi_f, \xi_a)\, \Theta_{ai}(\xi_a, \xi_i)\delta_{\nu_a \nu_i} .$$

From (5.17), by the use of (3.30), we obtain the energy behaviour of the various elements of the S-matrix (for $f \neq a \neq i$) at the threshold for channel a. Writing in terms of the $\mathcal{M}$-matrix defined by (4.5), we have

$$(5.22) \qquad \frac{\partial}{\partial |k_a|} \mathcal{M}^J_{fl_f s_f, i l_i s_i} = -\frac{1}{2}\binom{1}{i} \mathcal{M}^J_{fl_f s_f, a0J}\, \mathcal{M}^J_{a0J, i l_i s_i} .$$

Also in (5.19), (5.20), (5.21), and (5.22) the derivative with respect to $|k_a|$ is finite, while the derivative with respect to any other channel momentum (for example k_i or k_f) is infinite both from above and from below the threshold (*). The measurement of the cross-sections (differential, integrated over angles, total), at the threshold of channel a, will exibit a characteristic « cusp » or « rounded step » as shown in Fig. 2.
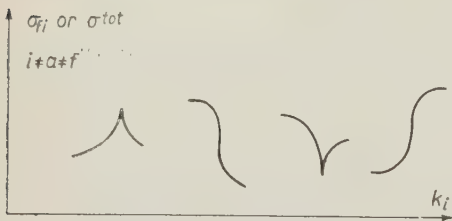

Fig. 2. - Various forms of cross-section behaviour at the threshold for channel a.

The downward cusp and the S-like step (3rd and 4th cases in Fig. 2) are not possible for the scattering cross-section when below the considered threshold there is only one channel (channel 1) open. To see this in detail we follow an argument given by NEWTON [5].

(*) It is clear from our method that the reason why the energy derivative is infinite also below the threshold for channel a is a purely quantum mechanical one. No explanation on classical grounds can be found for it as for the same phenomenon from above which can be interpreted on the base of the conservation of total flux see (also ref. [8]).

Let us consider for simplicity the case in which the threshold effect appears in the $l = 0$ wave of channel 1 and let us forget about the spins. We have for the $l = 0$ partial cross-section at threshold

$$(5.23) \qquad \frac{\partial}{\partial |k_a|} \sigma_{11}^{(0)} = \frac{\pi}{k_{10}^2} \frac{\partial}{\partial |k_a|} |1 - S_{11}|^2 ,$$

where S_{11} refers to $J = l_f = l_i = 0$ and k_{10} is k_1 evaluated at $k_a = 0$. Below the threshold for channel a, S_{11} is unitary and can be expressed as $\exp[2i\delta_{11}]$ with δ_{11} real number. Above the threshold the scattering phase shift δ_{11} becomes complex, but since $|S_{11}|^2 \leqslant 1$ from unitarity, its imaginary part must be non-negative. We have therefore near the threshold

$$(5.24) \qquad \delta_{11} = \delta_{110} + \begin{pmatrix} i \\ -1 \end{pmatrix} \alpha |k_a| ,$$

where we have neglected terms in k_a^2, k_a^3, etc.; $\alpha \geqslant 0$ and δ_{110} is δ_{11} evaluated at $k_a = 0$. By the use of (5.24) we easily evaluate eq. (5.23) above and below the threshold for channel a. In the limit as $k_a \to 0$, we get

$$(5.25a) \qquad \frac{\partial}{\partial k_a} \sigma_{11}^{(0)} \bigg|_a = -\frac{8\pi}{k_{10}^2} \alpha \sin^2 \delta_{110} \qquad\qquad \text{above} ,$$

$$(5.25b) \qquad \frac{\partial}{\partial |k_a|} \sigma_{11}^{(0)} \bigg|_b = -\frac{8\pi}{k_{10}^2} \alpha \sin \delta_{110} \cos \delta_{110} \qquad\qquad \text{below} .$$

Eq. (5.25a) gives us the desired result that above the threshold the derivative of $\sigma_{11}^{(0)}$ is always negative.

The ratio of (5.25a) to (5.25b) gives

$$(5.26) \qquad \frac{\partial/\partial k_a \, \sigma_{11}^{(0)} |_a}{\partial/\partial |k_a| \, \sigma_{11}^{(0)} |_b} = \mathrm{tg}\, \delta_{110} .$$

Because of (5.18) we have that if $\mathrm{tg}\, \delta_{110}$ is positive then the threshold anomaly is a cusp (1st case in Fig. 2), if $\mathrm{tg}\, \delta_{110}$ is negative then the anomaly is a rounded step (2nd case in Fig. 2) (*).

Eq. (5.25) enables us to obtain the scattering phase shift δ_{11} at the threshold for channel a by the measurement of the slope of $\sigma_{11}^{(0)}$ above and below the threshold in the experimental curve.

Various other examples in which the direct use of eq. (5.19) and (5.20) furnishes information about scattering phase shifts, have been considered in detail by BAZ [4] and NEWTON [6].

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(*) An equation similar to (5.26) holds true for any scattering cross section, at the threshold of a generic channel a, in the perturbation approximation which assumes the off-diagonal potentials to be weak; see equation (5.9) of reference [5].

It is now clear how a phenomenon of this kind can give information about relative parities and spins. In fact, only the $l_a = 0$ part of the wave in channel a contributes to (5.19), (5.20) and (5.21). Therefore, if it is possible to tell experimentally in which partial wave, in the initial or in the final channel, the cusp (or step) appears, the parity or spin of that channel will result from conservation of total angular momentum and parity. For example: if one discovers that in the cross-section σ_{fi} the cusp appears for $l_f = 0$, then $P_{af} = +1$, P_{af} being the relative parity between channel a and channel f. If instead the cusp appears for $l_f = 1$, then $P_{af} = -1$. This kind of analysis can be profitably applied, for example, to strange particle processes. The example just given works beautifully for $i = \pi^- p$, $f = \Lambda K$, $a = \Sigma K$, (see ref. ([11,12,14])) (*). The experimental determination of the final partial wave in which the cusp appears for the reaction π p $\rightarrow \Lambda K$ at the threshold for Σ production, is able to determine the $P_{\Lambda\Sigma}$ relative parity. And, for $P_{\Lambda\Sigma} = +1$, only a spin $\frac{1}{2}$ for Σ is compatible with a spin $\frac{1}{2}$ of Λ (spin of the K-meson taken $= 0$) (**).

Threshold effects can be also useful for the determination of the reaction cross-section σ_{fa} which in many instances is out of reach of direct experimental measurement. For the case in which one of the particles in channel a has spin zero so that s_a is given, taking the square of (5.21) above and below we get at threshold ([14])

$$(5.27) \qquad \left(\frac{\partial}{\partial k_a} \sigma_{fi}^{(l_f)} \Big|_a \right)^2 + \left(\frac{\partial}{\partial |k_a|} \sigma_{fi}^{(l_f)} \Big|_b \right)^2 = \pi^{-1} \sigma_{fi}^{(l_f)} (k_a \sigma_{fa}^{(l_f)}) (k_a^{-1} \sigma_{ai}^{(l_a=0)}) ,$$

where it is understood that $\nu_a = \nu_i$. Since $\sigma_{fa}^{(l_f)}$ goes to infinity linearly in the limit $k_a \rightarrow 0$, a k_a has been placed to counterbalance this effect. Analogously $\sigma_{ai}^{(l_a=0)}$ goes to zero linearly and k_a^{-1} makes it non-zero in the limit. Clearly only l_f-waves which are coupled to the $l_a = 0$ wave satisfy (5.27). From measurements of the slope of $\sigma_{fi}^{(l_f)}$ above and below the threshold and of $(k_a^{-1} \sigma_{ai}^{(l_a=0)})$, one can determine the $\sigma_{fa}^{(l_f)}$ cross-section at zero energy. If its right-hand side is known, (5.27) can be used to predict the size of the cusp (or rounded step) since near the threshold we can write: $\sigma_{fi} = \sigma_{fi}|_{k_a=0} + k_a (\partial \sigma_{fi} / \partial k_a)|_{k_a=0}$, neglecting powers of k_a greater than the first.

Eq. (5.17) does not hold if the final channel is the new channel a. In fact, one can easily see that for $f = a$ an extra term appears at the right hand side due to the derivative of $\varphi_v^{(a)}$ in the transition amplitude. This term is

(*) See also the recent paper by J. SUCHER, G. A. SNOW and T. B. DAY: Phys. Rev., 122, 1645 (1961).

(**) Experimental work is now in progress: F. EISLER, P. FRANZINI, J. M. GAILLARD, A. GARFINKEL, J. KEREN, R. PLANO, A. PRODELL and M. SCHWARTZ: Rev. Mod. Phys., 33, 436 (1961); S. E. WOLF, N. SCHMITZ, L. J. LLOYD, W. LASKAR, F. S. CRAWFORD jr., J. BUTTON, J. A. ANDERSON and G. ALEXANDER: Rev. Mod. Phys., 33, 439 (1961).

proportional to the $l_a = 1$ wave in channel a, so that it gives no contribution when consideration is given to cross-sections integrated over angles. The analogous of eq. (5.21) then yields[6]

$$(5.28) \qquad \frac{\partial}{\partial k_a}\left(k_a^{-1}\sigma_{ai}^{(l_a=0)}\right) = -\frac{1}{2\pi}\left(k_a\sigma_a^{tot}\right)\left(k_a^{-1}\sigma_{ai}^{(l_a=0)}\right),$$

where $\nu_a - \nu_i$. (5.28) allows the determination of σ_a^{tot} from the slope and intercept of the curve $(k_a^{-1}\sigma_{ai}^{(l_a=0)})$ near the threshold.

The same procedure applied to the case $f = i = a$ gives no more than the known zero energy property:

$$(5.29) \qquad \frac{\partial}{\partial k_a}\sigma_{aa} \leqslant 0,$$

where the equality holds when below the threshold for channel a there is no channel open. For the example considered above of production of strange particles in pion-nucleon collisions, by using eq. (5.27) one can determine, at zero energy, the cross-section $\sigma_{\Sigma K \to \Lambda K}^{(l_{\Sigma}=0)}$ for the process $\Sigma K \to \Lambda K$ which is not feasible experimentally. Analogously, use of (5.28) at the ΛK and ΣK thresholds yields the non directly measurable cross-sections $\sigma_{\Lambda K}^{tot}$ and $\sigma_{\Sigma K}^{tot}$, respectively.

Threshold anomalies can also give information about the existence of new particles. For example, the discovery of a narrow singularity in the energy dependence of the cross-sections for the processes $\pi^- + p \to \pi - p$ and $\pi^- + p \to \to \pi^0 + n$ would support the existence of the reaction $\pi^- + p \to \rho^0 + n$ with production of the ρ^0 meson [31,32] (*). On the same line of thought, proton-proton collisions, and elastic neutron scattering by nuclei near the threshold for the (n, 2n) reaction, give information about a « pion +nucleon » bound system [33] and about the dineutron [34], respectively.

The threshold anomalies considered in this section have been observed experimentally in the elastic scattering $^3H(p, p)^3H$ at the threshold for the reaction $^3H(p, n)^3He$ in the form of a downward cusp [35], and in the scattering Li(p, p)Li at the threshold for Li(p, n)Be in the form of a cusp [36].

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[31] V. G. ZINOV, A. D. KONIN, S. M. KORENCHENKO and B. PONTECORVO: *Žurn. Éksp. Teor. Fiz.*, **36**, 1948 (1959); translation: *Soviet Phys. JETP*, **9**, 1386 (1959).

[32] V. I. GOLDANSKI and YA. A. SMORODINSKI: *Žurn. Éksp. Teor. Fiz.*, **36**, 1950 (1959); translation: *Soviet Phys. JETP*, **9**, 1387 (1959).

(*) The equation for the cross-sections which appears in [32]. can be obtained from our (5.27).

[33] YU. D. PROKOSHKIN, V. I. RYKALIN and I. M. VASILEVSKI: Report D-678, Dubna (February 1961).

[34] A. I. BAZ and YA. A. SMORODINSKI: *Compt. Rend. du Congrès International de Physique Nucléaire* (Paris, 1958), p. 579.

[35] M. E. ENNIS and A. HEMMENDINGER: *Phys. Rev.*, **95**, 772 (1954).

[36] P. R. MALMBERG: *Phys. Rev.*, **101**, 114 (1956).

We end this section with a brief discussion of the case in which three particles are present in the new channel a (e.g., photoproduction of two pions on hydrogen, production of an extra nucleon in nucleon-nucleus collisions, etc.).

The quantity which exerts an influence on the scattering and reaction cross-sections σ_{fi} via removal of flux, is now the reaction cross-section $\bar{\sigma}_{ai}$ integrated over the energy range available to one of the particles in channel a: $\int_0^{k_1^{\max}} \mathrm{d}^3k_1\bar{\sigma}_{ai}$. This quantity (see (4.14)) and its first derivative with respect to k_i are zero at threshold so that no threshold effect of the kind of a cusp (or rounded step) is expected in σ_{fi}. Since the second derivative is different from zero and finite, a discontinuity is instead expected in the second derivative of σ_{fi} with respect to k_i at threshold. Such expectations are confirmed by a quantitative analysis, on the same line as before, that we omit. The quantity that plays now the decisive role is the free three-particle Green's function for channel a. The effect predicted is, however, not useful (*).

Special attention must be paid instead to the case in which there are (at least) two different reactions with three particles in the final state. If one of the final particles (α) is common to both the considered reactions, then a threshold effect is expected in the spectrum of the particle α coming from the lighter three-body channel due to (4.13) (with $f - a$). The effect appears as a function of the momentum of the particle α while the total energy is fixed and lends itself useful for the determination of relative parities and spins of the reaction products, and of cross-sections for processes not feasible experimentally. Details can be found in ref. ([17-19]) (**).

6. – Coulomb effects.

From Fig. 1 we see that when a repulsive Coulomb force is present in channel a, then σ_{ai} starts at the threshold like $\exp[-\pi A_a/k_a]$, while in the case of Coulomb attraction in a, σ_{ai} starts with a finite value at the threshold and in general with non-zero and finite slope. We expect, therefore, that in the case of Coulomb repulsion the new channel a makes itself felt in the other cross-sections very smoothly and no cusp (or rounded step) will consequently appear in them ([2,7]). Also in the case of Coulomb attraction no cusp is expected, but instead another type of anomaly is to be observed. As the energy of the

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(*) The size of the discontinuity in the second derivative of σ_{fi} with respect to k_i has been given by L. M. DELVES: n-d *elastic and inelastic scattering near the inelastic threshold* (preprint, M.I.T., 1961).

(**) The method discussed in this section is also applicable in this case with some changes.

incident particle approaches the threshold for channel a from below, we will observe in σ_{fi} a series of ever more rapid oscillations due to the physical possibility of excitation of the infinitely many Coulomb bound states of channel a. The limit of σ_{fi} below the threshold will not exist, since these Coulomb bound states, and therefore the corresponding resonances in σ_{fi}, have the threshold energy as an accumulation point. What will be seen experimentally is the average of σ_{fi} below the threshold of channel a, and this quantity will eventually exhibit a step-like discontinuity at threshold, to counterbalance the sudden leakage of flux to the new channel a [7].

The Coulomb effects are able therefore, in both the repulsive and in the attractive case, to wash out the cusp (or rounded step), as given in Fig. 2, in the old cross-sections at the threshold for the new channel a. In most cases of interest, however, such as in processes in which pion and/or strange particles take place, the Coulomb effects described above cover an energetic region around the threshold of the order of 10^{-3} MeV, which is out of reach to the present techniques due to the poor energy resolution in the incoming beam. In these cases the overall energy behaviour of the various σ_{fi} at the threshold for channel a *will look like a cusp* (or a rounded step) even though σ_{fi} does *not* exibit infinite derivatives at either side of the threshold in either case (repulsive or attractive). In these cases therefore arguments like the ones given before on the determination of parity, spins, etc., will still be applicable.

To put into a quantitative form the above considerations, we start again with taking into account eq. (5.3) for the complete Green's function. Since we are interested in a reaction for which both the initial and the final channels are different from the newly opened channel a, the threshold anomalies will be given exclusively by the behaviour of $\mathcal{G}$. We define for a function $f(k_a)$

$$(6.1) \qquad \Delta f(k_a) = \underset{\bar{k}_a \to 0^+}{\text{Lim}} \, f(\bar{k}_a) - f(k_a) ,$$

with k_a taken below and near the threshold for channel a. Then, for the complete Green's function we have

$$(6.2) \qquad \Delta \mathcal{G} = (1 + G_b V) P_a \Delta \mathcal{G}_a P_a (1 + V G_b)$$

since G_b does not « see » channel a.

In order to evaluate $\Delta \mathcal{G}_a$ we write for $\mathcal{G}_a$ the integral equation

$$(6.3) \qquad \mathcal{G}_a = G_{ca} + G_{ca}(V'_{aa} + V'_{ab} G_b V'_{ba}) \mathcal{G}_a .$$

Taking again into consideration the fact that we want only outgoing waves

into $\Delta\mathscr{G}_a$, we get

(6.4) $\quad \Delta\mathscr{G}_a = [1 + (\Delta G_{ca} - G_{ca})(V'_{aa} + V'_{ab}G_b V'_{ba})]^{-1}\Delta G_{ca}[1 + (V'_{aa} + V'_{ab}G_b V'_{ba})\mathscr{G}_a] \equiv$

$\qquad [1 + \mathscr{G}_a(V'_{aa} + V'_{ab}G_b V'_{ba})] \cdot \{[\Delta G_{ca}]^{-1} + T'_{aa}\}^{-1}[1 + (V'_{aa} + V'_{ab}G_b V'_{ba})\mathscr{G}_a],$

where with the operator T'_{fi} we mean

(6.5) $\qquad\qquad\qquad T'_{fi} = V'_{fi} + \sum_{j,k} V'_{fj}\mathfrak{G}_{jk}V'_{ki},$

which is so defined that

(6.6) $\qquad\qquad T_{fi} = T_{ci}\delta_{fi}\delta_{s_f s_i}\delta_{v_f v_i} + (\varphi_c^{(f)(-)}, T'_{fi}\varphi_c^{(i)(+)}).$

All the quantities on the right-hand side of (6.4) are defined in the limit as the threshold energy for channel a is reached from above. Substituting (6.4) into (6.2) and using (5.3), one gets the following equation for Θ_{fi}:

(6.7) $\qquad \Delta\Theta_{fi}(\boldsymbol{\xi}_f, \boldsymbol{\xi}_i) = -\left(\frac{2\pi}{\hbar}\right)^2 (\mu_i\mu_f k_f/k_i)^{\frac{1}{2}}\left(\varphi_c^{(f)(-)}, T'_{fa}\frac{1}{[\Delta G_{ca}]^{-1} + T'_{aa}}T'_{ai}\varphi_c^{(i)(+)}\right).$

Obviously, there is no effect in Θ_{ci} since this quantity is diagonal on the channel index. We see, therefore, that the anomaly in Θ_{fi} comes exclusively from G_{ca}.

By using eq. (3.14), we can evaluate explicitly the jump experienced by the Coulomb Green's function G_{ca}. The quantity which is responsible for such a jump across the threshold is

(6.8) $\qquad\qquad (-ik_a)^l\Gamma(l+1+i\eta_a)W_{-i\eta_a,\,l+\frac{1}{2}}(-2ik_a r_>),$

while at the threshold $M_{i\eta_a,\,l+\frac{1}{2}}(2ik_a r_<)/(2ik_a r_<)^{l+1}$ is a continuous function of the energy. The quantity (6.8) has been considered in great detail in ref. ([7]). From eq. (16) of that paper we have

(6.9) $\qquad \Delta[(-ik_a)^l\Gamma(l+1+i\eta_a)W_{-i\eta_a,\,l+\frac{1}{2}}(-2ik_a r_>)] =$

$\qquad = \begin{cases} 0, & \text{for } A_a > 0, \\[2ex] \dfrac{2\pi i}{1-\exp[-2i\pi|\eta_a|]}\dfrac{|A_a|^{2l+1}}{(2l+1)!}\dfrac{M_{i|\eta_a|,\,l+\frac{1}{2}}(2i|k_a|r_>)}{(i|k_a|)^{l+1}}, & \text{for } A_a < 0. \end{cases}$

For the Coulomb Green's function we get

(6.10) $\Delta\langle\boldsymbol{r}|G_{ca}|\boldsymbol{r}'\rangle = \begin{cases} 0, & \text{for } A_a > 0, \\[2ex] -\dfrac{i\mu_a|k_a|}{\pi\hbar^2}(1-\exp[-2i\pi|\eta_a|])^{-1}\displaystyle\sum_l (2l+1)P_l\left(\dfrac{\boldsymbol{r}\cdot\boldsymbol{r}'}{rr'}\right)\cdot \\ \qquad\cdot\dfrac{F_l(|k_a|r_<)}{|k_a|r_<}\dfrac{F_l(|k_a|r_>)}{|k_a|r_>}, & \text{for } A_a < 0. \end{cases}$

As expected, we do not get any effect in G_{ca}, and therefore in Θ_{fi}, in the repulsive case. For attractive Coulomb forces in channel a we obtain instead a jump in G_{ca}, and in Θ_{fi}, when we go across the threshold. In the following we will therefore continue to discuss the attractive case.

Substitution of (6.10) into (6.7) yields a simple expression for $\Delta\Theta_{fi}$ only if we expand in spherical harmonics:

$$(6.11) \qquad \Delta\Theta_{fi}(\boldsymbol{\xi}_f, \boldsymbol{\xi}_i) = -\frac{2\pi i}{k_i} \sum_{\substack{JMl_il_f \\ l_as_al_as_a}} Y_{l_f}^{M-\nu_f}(\boldsymbol{k}_f)\, Y_{l_i}^{M-\nu_i*}(\boldsymbol{k}_i) \cdot$$

$$\cdot\, C_{l_fs_f}(J, M;\ M - \nu_f, \nu_f)\, C_{l_is_i}(J, M;\ M - \nu_i, \nu_i)\, S_{fl_fs_f,al_as_a}^{J} \cdot$$

$$\cdot \left(\frac{1}{S_a^J - \exp\left[2i(\sigma_l^a - \pi\,|\eta_a|)\right]}\right)_{al_as_a, al_as_a'} S_{al_as_a', il_is_i}^{J},$$

where S_a^J is the submatrix whose elements refer only to channel a. $\exp\left[2i\sigma_l^a\right]$ is a diagonal matrix in the l-representation for channel a.

If we renormalize the definition of the S-matrix by

$$(6.12) \qquad \overline{S}_{jk} = \exp\left[-i\sigma_0^j\right] S_{jk} \exp\left[-i\sigma_0^k\right],$$

which does not alter any of the observable quantities, and taking into account the fact that

$$(6.13) \qquad \lim_{k_a \to 0} \exp\left[2i(\sigma_l - \sigma_0)\right] = \exp\left[i\pi l\right],$$

we obtain an expression for $\Delta\overline{\Theta}_{fi} = \exp\left[-i\sigma_0^f\right]\Delta\Theta_{fi}\exp\left[-i\sigma_0^i\right]$ in which only $\overline{S}$-matrices appear and the $\exp\left[2i(\sigma_l^a - \pi\,|\eta_a|)\right]$ is substituted with $\exp\left[i\pi(l - 2\,|\eta_a|)\right]$; again, $\exp\left[i\pi l\right]$ is a matrix in the l-representation for channel a. $\Delta\overline{\Theta}_{fi}$ is more suitable for applications in that it does not contain the unobservable oscillations at $k_a = 0$ which are present in (6.11) due to $\exp\left[2i\sigma_l^a\right]$.

In terms of the $\overline{S}$-matrices we get from (3.28)

$$(6.14) \qquad \Delta\sigma_i^{\text{tot}}(\boldsymbol{\xi}_i) = -\frac{2\pi}{k_i^2}\,\text{Re}\sum_{\substack{Jl_il_f \\ l_as_al_as_i'}} \sqrt{(2l_i+1)(2l_f+1)} \cdot$$

$$\cdot\, C_{l_fs_i}(J, \nu_i;\ 0, \nu_i)\, C_{l_is_i}(J, \nu_i;\ 0, \nu_i)\, \overline{S}_{il_fs_f,al_as_a}^{J} \cdot \left(\frac{1}{\overline{S}_a^J - \exp\left[i\pi(l - 2\,|\eta_a|)\right]}\right)_{al_as_a, al_as_a'} \overline{S}_{al_as_a', il_is_i}^{J} \cdot$$

From (6.11) and (3.30) we obtain the energy behaviour of the elements of the $\overline{S}$-matrix (for $f \neq a \neq i$) at the threshold for channel a:

$$(6.15) \qquad \Delta\overline{S}_{fl_fs_f, il_is_i}^{J} = \sum_{l_as_al_as_a'} \overline{S}_{fl_fs_f, al_as_a}^{J} \left(\frac{1}{\overline{S}_a^J - \exp\left[i\pi(l - 2\,|\eta_a|)\right]}\right)_{al_as_a, al_as_a'} \overline{S}_{al_as_a', il_is_i}^{J} \cdot$$

The oscillatory denominator in (6.11), (6.14) and (6.15) represents the resonances at each of the infinitely many Coulomb bound states of the channel a. These oscillations become infinitely fast near $k_a = 0^-$, point of accumulation for the Coulomb bound states. Therefore only the average of the various observable quantities will be detected experimentally. The averaging procedures for the differential reaction cross-sections and for the total cross-section have been given in ref. ([9]). The result is that, while the various averaged differential cross-sections exhibit a jump across the threshold for channel a, the averaged total cross-section turns out to be continuous there.

A phenomenon of this kind is for example expected to occur in the elastic scattering of X-rays by atoms at the threshold for the photoelectric effect ([9]). There the step-like discontinuity in the scattering cross-section is of the same order of magnitude as the cross-section itself and is equal and opposite to the initial step in the σ-photoeffect. Fig. 3 shows the details of the phenomenon. The effect has never been observed up to now since the experiments have been set up either to measure the index of refraction that, when measured directly, does not show the anomaly, or the mass absorption coefficient which does not exhibit the anomaly either.
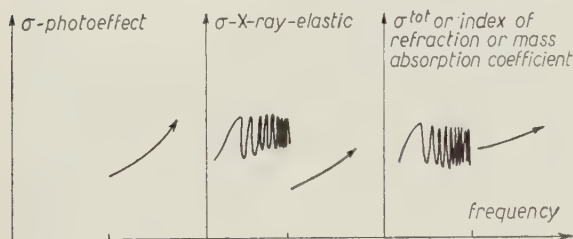
Fig. 3. – Photoeffect.

At the threshold for a three-body channel a, from the considerations at the end of Section **4**, no effect is expected to occur in the case of Coulomb repulsion. If two of the particles in channel a are attractively charged, then an anomaly should be observed in the second derivative of the old cross-sections at the threshold for channel a. Coulomb effects of the kind discussed in this section are expected to occur also in the case mentioned at the end of Section **5**.

The method used in this paper for the discussion of the threshold anomalies applies in a straightforward way when Coulomb forces are present in the newly opened channel a. This is the reason why, for uniformity of treatment, it has been preferred over other methods which are able to reach the conclusions of Section **5** by following a different line of thought.

Beside the well known $\mathscr{R}$-matrix approach employed by WIGNER ([1]) and BREIT ([2,3]), essentially two other methods can be found in the literature: the one by BAZ abd OKUN ([12]) based on the unitarity on the $S^J_{fl_f s_f, al_a s_a}$ matrix used above and below the threshold for the new channel a, and the one by NEWTON ([6]) based on a K-matrix formalism. These two methods are not, however, easily extensible to the Coulomb case.

* * *

Self-Learning in PAPA.

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The PAPA machine [1] can learn by itself without teacher, *i.e.* the machine is able to organize its experimental data into classes without human intervention. Here is how it operates.

The machine separates at random the N objects (examples) shown to it into two sets of $N/2$ objects—called class a and class b respectively—and learns these two classes in the usual way, as if they were really two objectively different classes taught to it by a human teacher. After this, the same N objects are shown again to the machine for recognition. PAPA will then recognize N' objects as belonging to class a, and $N-N'$ objects as belonging to class b. Of course the N' objects assigned to class a will *not* in general be those $N/2$ objects belonging to the original class a, since PAPA does make mistakes with a finite number of A units. The reason for this is as follows.

In the original class a, due to random fluctuations, there might be a higher percentage than in class b of objects satisfying a certain property. For example, let the objects of class a be on the average of greater size than those of class b. This introduce a bias [2] in that, other things being equal, there is a greater chance that PAPA will make mistakes when a *large* object of the original class b or a *small* object of class a is presented for recognition. It is now clear that the average size of the N' objects recognized as belonging to class a will be even greater than it was in the original class. Similarly the $N-N'$ objects,

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[1] A. Gamba: *Proc. IRE*, **49**, 349 (1961); G. Palmieri and A. Sanna: *Methodos*, to be published; A. Gamba, L. Gamberini, G. Palmieri and R. Sanna: *Suppl. Nuovo Cimento*, **20**, 112 (1961).

[2] A bias from our point of view: a perfectly valid criterion from the point of view of PAPA.

recognized as belonging to class b will now have smaller average sizes. We can now define two new classes a' and b' made up by the N' and $N-N'$ objects recognized as a and b respectively, and start the process over again. A stage will be reached when the objects will be separated according to their sizes, or to the property that originally fluctuated the most. PAPA operates therefore as a truly « intelligence amplifier ».

The process can be applied separately on each of the two classes (large and small objects) in order to obtain a further classification into sub-classes, etc. It is clear that the complete ordering of objects obtained in this way is not necessarily the one a human would do, although it satisfies some of the more obvious requirements of a man-made classification. For example, the first dichotomic separation into two classes is based on « the most widely fluctuating property », which is also in most cases the most evident or simple property for the sense-organs of the machine. Therefore the machine starts from its « most simple and evident » properties and proceeds to the more sophisticated ones using a « criterion of simplicity ».

Experiments on this type of learning are now in progress and will be reported later.

A Generalized Hamiltonian Dynamics
for Relativistic Particles with Spin - II.

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CONTENTS. — **1**. Introduction. – **2**. The dynamics scheme relative to the Frenet frame. – **3**. Some geometrical results. – **4**. The equivalent two mass-point representation. – **5**. Introduction of internal co-ordinates and their conjugate momenta. – **6**. Concluding remarks.

1. – Introduction.

In a previous paper (¹) (henceforward referred to as I) the motion of a finite relativistic free particle with spin was discussed in terms of the co-ordinates x_α ($\alpha = 1, 2, 3, 4$) of some point P within the body and the direction cosines U^μ_α ($\mu = 0, 1, 2, 3$) of a tetrad of orthogonal vectors forming a frame of reference moving with the particle. A generalized Hamiltonian dynamics was proposed for the particle by defining Poisson brackets (P.b.), the Hamiltonian, and the equations of motion. It was shown that the dynamics is invariant to an orthogonal transformation of the U^μ_α; and that in particular for the U^μ_α could be taken quantities V^μ_α which satisfy the Frenet equations in four dimensions. The frame of reference, whose vectors have for direction cosines the quantities V^μ_α will be called the Frenet frame.

It was postulated that the free particle had a conserved momentum p_α and total angular momentum $M_{\alpha\beta}$, the latter being the sum of the external

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(¹) J. B. HUGHES: *Suppl. Nuovo Cimento*, **20**, 89 (1961).

angular momentum (or moment of momentum) $L_{\lambda\beta}$ and the spin angular momentum $S_{\lambda\beta}$. The momentum was defined as a linear function of the $U^\mu{}_\lambda$; the spin angular momentum, a quadratic function of the $U^\mu{}_\lambda$. By requiring that certain P.b. relations should be satisfied (*) the coefficients appearing in the definition of p_λ and $S_{\lambda\beta}$ were uniquely defined in terms of quantities $\Omega_{\mu\nu}$ which appeared in the equations of motion for the $U^\mu{}_\lambda$. It was assumed that, for a free particle, these coefficients are independent of position along the world line of P.

2. – The dynamics scheme relative to the Frenet frame.

The dynamical variables are x_α and $V^\mu{}_\lambda$, where $V^0{}_\alpha$, $V^1{}_\alpha$, $V^2{}_\alpha$, $V^3{}_\alpha$ are the direction-cosines of the tangent, normal, binormal, and trinormal, respectively, of the world line of P. Let ξ be the arc length measured along the world line of P, and for any quantity $u = u(x_\alpha, V^\mu{}_\alpha)$ let $\dot u \equiv du/d\xi$.

Then

$$(2.1) \qquad \dot x_\alpha = V^0{}_\alpha,$$

$$(2.2) \qquad \dot V^\mu{}_\alpha = \Omega_{\mu\nu} V^\nu{}_\alpha,$$

where the quantities $\Omega_{\mu\nu}$ are given by the matrix

$$(2.3) \qquad \Omega = \begin{bmatrix} 0 & \dfrac{1}{\varrho} & 0 & 0 \\[2mm] -\dfrac{1}{\varrho} & 0 & \dfrac{1}{\sigma} & 0 \\[2mm] 0 & -\dfrac{1}{\sigma} & 0 & 0 \\[2mm] 0 & 0 & 0 & 0 \end{bmatrix},$$

and $1/\varrho$ is the curvature, $1/\sigma$ the torsion of the world line of P, ϱ and σ being constant. From (2.2) and (2.3) we see that the world line of P has zero tilt.

The Hamiltonian is defined by

$$(2.4) \qquad H = \frac{m}{2} V^\mu{}_\alpha V^\mu{}_\alpha.$$

Poisson brackets by

$$(2.5) \qquad m(u, v) = \frac{\partial u}{\partial x_\alpha} \frac{\partial v}{\partial V^0{}_\alpha} - \frac{\partial u}{\partial V^0{}_\alpha} \frac{\partial v}{\partial x_\alpha} + \Omega_{\mu\nu} \frac{\partial u}{\partial V^\mu{}_\alpha} \frac{\partial v}{\partial V^\nu{}_\alpha},$$

(*) They were that (x_α, x_β); (p_α, p_β); (x_α, p_β); $(M_{\alpha\beta}, M_{\mu\nu})$ should have the same values as in classical point mechanics.

—where u and v are any functions of the dynamical variables—and the general equation of motion is

$$(2.6) \qquad \dot{u} = \frac{\partial u}{\partial \xi} + (u, H) \ .$$

From I (eq. (5.1), (5.3), (6.23), (6.24) and (9.8)) the momentum p_α and spin angular momentum $S_{\alpha\beta}$ are given by

$$(2.7) \qquad p_\alpha = m V^0_{\ \alpha} + m \frac{\sigma}{\varrho} V^2_{\ \alpha} \ ,$$

$$(2.8) \qquad S_{\alpha\beta} = m\sigma(V^1_{\ \alpha} V^2_{\ \beta} - V^2_{\ \alpha} V^1_{\ \beta}) \ ,$$

and the spin pseudovector s_α is given by

$$(2.9) \qquad s_\mu = m\sigma\varepsilon_{\mu\nu\lambda\beta} V^1_{\ \alpha} V^2_{\ \beta} V^0_{\ \nu} = \varepsilon m \sigma V^3_{\ \mu} \ ,$$

where ε is a pseudoscalar of magnitude unity.

The equation of motion for the point P $\big(\text{I } (7.2)\big)$ becomes

$$(2.10) \qquad \dot{x}_\alpha = A_\alpha \cos(\lambda\xi) + B_\alpha \sin(\lambda\xi) + \frac{p_\alpha}{m}\frac{1}{\sigma^2}\left(\frac{1}{\sigma^2} + \frac{1}{\varrho^2}\right)^{-1} = $$

$$= A_\alpha \cos(\lambda\xi) + B_\alpha \sin \lambda\xi + \frac{p_\alpha}{m}\left(\frac{\varrho^2}{\sigma^2 + \varrho^2}\right) \ ,$$

where

$$(2.11) \qquad \lambda^2 = \frac{1}{\varrho^2} + \frac{1}{\sigma^2} \ .$$

Clearly it is the sum of an uniform rectilinear motion and an oscillatory motion.

3. – Some geometrical results (*).

The Frenet formulae show that the world line of P is a curve whose curvature and torsion are constant and whose tilt is zero. Since the tilt is zero it is a 3-dimensional curve and lies in a flat, called the oscillating flat, to which the trinormal V^3 is perpendicular (§§ 128, 139, 146, 166). Hence the spin pseudovector is orthogonal to the flat in which the world line of P lies.

(*) These come from the indicated paragraphs of A. R. FORSYTH: *Geometry of Four Dimensions*, (Cambridge, 1930).

The world line of P, being a curve with constant curvature and torsion, and zero tilt, is a geodesic in a sphero-cylindrical region and is the extension to four-space of the cylindrical helix in three-space (§ 302, Ex. 2).

When there is no tilt, the infinitesimal transformation connecting the V^μ_λ at consecutive points on the world line of P can be effected, in the flat to which V^3 is perpendicular, by a rotation round the line $V^2_\lambda = (\sigma/\varrho)V^0_\lambda$ in the plane $V^1_\alpha = 0$ (§ 166).

The quantity λ appearing in eq. (2.10), and which gives the frequency of the oscillatory part of the motion of P, satisfies the equation

$$\lambda^2 = \frac{1}{\varrho^2} + \frac{1}{\sigma^2} = \frac{\mathrm{d}\iota}{\mathrm{d}\xi},$$

where ι is the inclination of two consecutive vectors in the direction of V^1_x (§ 131).

4. – The equivalent two mass-point representation.

Eq. (2.2) with $\mu = 1$ gives

(4.1) $$\dot{V}^1_\alpha = -\frac{1}{\varrho}V^0_\alpha + \frac{1}{\sigma}V^2_\alpha,$$

so that

$$V^2_\alpha = \sigma\dot{V}^1_\alpha + \frac{\sigma}{\varrho}V^0_\alpha.$$

Therefore (2.7) can be written

$$p_\lambda = mV^0_\alpha + m\frac{\sigma}{\varrho}\left|\sigma\dot{V}^1_\lambda + \frac{\sigma}{\varrho}V^0_\lambda\right|,$$

$$= mV^0_\alpha + m\frac{\sigma^2}{\varrho^2}[V^0_\alpha + \varrho\dot{V}^1_\alpha]$$

$$= m\frac{\mathrm{d}x_\alpha}{\mathrm{d}\xi} + m\frac{\sigma^2}{\varrho^2}\frac{\mathrm{d}}{\mathrm{d}\xi}(x_\alpha + \varrho^2\ddot{x}_\alpha),$$

or

(4.2) $$p_x = m\frac{\mathrm{d}x_\alpha}{\mathrm{d}\xi} + m'\frac{\mathrm{d}y_\alpha}{\mathrm{d}\xi},$$

where

(4.3) $$y_\alpha = x_\alpha + \varrho^2\ddot{x}_\alpha = x_\alpha + \varrho V^1_\alpha,$$

and

$$(4.4) \qquad\qquad m' = m\, \frac{\sigma^2}{\varrho^2}\,.$$

We will call the point, whose co-ordinates are y_α, Q.

Expression (4.3) for p_α, the momentum, could be regarded as the sum of the momenta of two point particles, one of mass m whose co-ordinates are x_α and the other mass m' whose co-ordinates are y_α.

Consider now the expression

$$(y_\alpha - x_\alpha)m' \frac{\mathrm{d}y_\beta}{\mathrm{d}\xi} - (y_\beta - x_\beta)m' \frac{\mathrm{d}y_x}{\mathrm{d}\xi} =$$

$$= \varrho^2 \ddot{x}_\alpha m \frac{\sigma^2}{\varrho^2}\left[V^0{}_\beta + \varrho\left(-\frac{1}{\varrho}V^0{}_\beta + \frac{1}{\sigma}V^2{}_\beta\right)\right] - \varrho^2 \ddot{x}_\beta m \frac{\sigma^2}{\varrho^2}\left[V^0{}_\alpha + \varrho\left(-\frac{1}{\varrho}V^0{}_\alpha + \frac{1}{\sigma}V^2{}_\alpha\right)\right] =$$

$$= m\sigma(V^1{}_\alpha V^2{}_\beta - V^2{}_\alpha V^1{}_\beta) = S_{\alpha\beta}\,.$$

Thus the expression for the spin angular momentum is equal to the expression for the moment about x_α of the momentum of the mass m' whose co-ordinates are y_α.

Moreover the point y_α is the centre of curvature, corresponding to the point x_α, of the world line of P (see FORSYTH, § 130).

5. – Introduction of internal co-ordinates and their conjugate momenta.

Let us now transform the quantities $V^\mu{}_\alpha$ to quantities $W^\mu{}_\alpha$ given by

$$(5.1) \qquad \begin{cases} W^0{}_\alpha = \dfrac{1}{\sigma\lambda}V^0{}_\alpha + \dfrac{1}{\varrho\lambda}V^2{}_\alpha\,, \\[2ex] W^1{}_\alpha = V^1{}_\alpha\,, \\[2ex] W^2{}_\alpha = -\dfrac{1}{\varrho\lambda}V^0{}_\alpha + \dfrac{1}{\sigma\lambda}V^2{}_\alpha\,, \\[2ex] W^3{}_\alpha = V^3{}_\alpha\,. \end{cases}$$

It is seen immediately that the vectors $W^\mu{}_\alpha$ are orthogonal

$$(5.2) \qquad\qquad W^\mu{}_\alpha W^\nu{}_\alpha = \delta^{\mu\nu}$$

and related to the $V^\mu{}_\alpha$ through an orthogonal transformation

$$(5.3) \qquad\qquad W^\mu{}_\alpha = O_{\mu\nu} V^\nu{}_\alpha\,.$$

The essential form of the dynamics is unchanged by this transformation but note that $\dot{x}_\alpha$ and $W^0{}_\alpha$ are not equal.

The derivatives of the $W^\mu{}_x$ with respect to ξ are

(5.4a) $$\dot{W}^0{}_\alpha = \quad 0 \,,$$

(5.4b) $$\dot{W}^1{}_\alpha = \quad \lambda W^2{}_\alpha \,,$$

(5.4c) $$\dot{W}^2{}_\alpha = -\,\lambda W^1{}_\alpha \,,$$

(5.4d) $$\dot{W}^3{}_\alpha = \quad 0 \,,$$

or

$$\dot{W}^\mu{}_\alpha = \Omega'_{\mu\nu} W^\nu{}_\alpha \,,$$

where

$$\Omega'_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$W^0{}_\alpha$ is a unit vector parallel to the total momentum p_α,

$W^1{}_\alpha$ is a unit vector parallel to PQ,

$W^2{}_x$ is a unit vector parallel to the velocity of the point Q relative to the point P,

$W^3{}_\alpha$ is a unit vector parallel to the constant spin pseudovector.

Using (4.3) and the result

$$\frac{\partial u}{\partial V^\sigma{}_\alpha} = O_{\alpha\sigma} \frac{\partial u}{\partial W^\mu{}_\alpha} \qquad (\mathrm{I}(10.2)) \,,$$

the Hamiltonian H is given by

(5.5) $$H = \frac{m}{2}\, W^\mu{}_\nu W^\mu{}_\nu \,,$$

and (P.b.) by

(5.6) $$m(u,\, v) = O_{\mu\sigma}\left(\frac{\partial u}{\partial x_\alpha} \frac{\partial v}{\partial W^\mu{}_x} - \frac{\partial v}{\partial x_\alpha} \frac{\partial u}{\partial W^\mu{}_x} \right) + \Omega'_{\mu\nu} \frac{\partial u}{\partial W^\mu{}_\alpha} \frac{\partial v}{\partial W^\nu{}_\alpha} \,,$$

where

(5.7) $$\Omega'_{\mu\nu} = O_{\mu\alpha} \Omega_{\alpha\beta} O_{\nu\beta} \,.$$

Thus

$$(5.8) \qquad m(u, r) = \frac{1}{\sigma\lambda}\left(\frac{\partial u}{\partial x_\alpha}\frac{\partial v}{\partial W^0{}_\alpha} - \frac{\partial r}{\partial x_\alpha}\frac{\partial u}{\partial W^0{}_\alpha}\right) + \frac{1}{\varrho\lambda}\left(\frac{\partial u}{\partial x_\alpha}\frac{\partial r}{\partial W^2{}_\alpha} - \frac{\partial v}{\partial x_\alpha}\frac{\partial u}{\partial W^2{}_\alpha}\right) +$$

$$+ \lambda\left(\frac{\partial u}{\partial W^1{}_\alpha}\frac{\partial v}{\partial W^2{}_\alpha} - \frac{\partial u}{\partial W^2{}_\alpha}\frac{\partial v}{\partial W^1{}_\alpha}\right).$$

Then

$$(5.9a) \qquad m(x_\alpha, W^0{}_\beta) = \frac{1}{\sigma\lambda}\,\delta_{\alpha\beta}\,,$$

$$(5.9b) \qquad m(x_\alpha, W^1{}_\beta) = 0\,,$$

$$(5.9c) \qquad m(x_\alpha, W^2{}_\beta) = \frac{1}{\varrho\lambda}\,\delta_{\alpha\beta}\,,$$

$$(5.9d) \qquad m(W^0{}_\alpha, W^1{}_\beta) = 0\,,$$

$$(5.9e) \qquad m(W^0{}_\alpha, W^2{}_\beta) = 0\,,$$

$$(5.9f) \qquad m(W^1{}_\alpha, W^2{}_\beta) = \lambda\delta_{\alpha\beta}\,.$$

If we define the point z_α by

$$(5.10) \qquad z_\alpha = x_\alpha + \frac{1}{\varrho\lambda^2}\,W^1{}_\alpha\,,$$

we find the following set of P.b. relations:

$$(5.11a) \qquad m(z_\alpha, W^0{}_\beta) = \frac{1}{\sigma\lambda}\,\delta_{\alpha\beta}\,,$$

$$(5.11b) \qquad m(z_\alpha, W^1{}_\beta) = 0\,,$$

$$(5.11c) \qquad m(z_\alpha, W^2{}_\beta) = 0\,,$$

$$(5.11d) \qquad m(W^0{}_\alpha, W^1{}_\beta) = 0\,,$$

$$(5.11e) \qquad m(W^0{}_\alpha, W^2{}_\beta) = 0\,,$$

$$(5.11f) \qquad m(W^1{}_\alpha, W^2{}_\beta) = \lambda\delta_{\alpha\beta}\,.$$

We have already seen that p_α can be considered as the sum of the momenta of a particle of mass m whose co-ordinates are x_α, and a particle of mass $m' = m(\sigma^2/\varrho^2)$ whose co-ordinates are $y_\alpha = x_\alpha + \varrho W^1{}_\alpha$. It is easily seen that z_α is the weighted mean of x_α and y_α with weights m and m'.

Moreover

$$(5.12) \qquad \dot{z}_\alpha = V^0{}_\alpha + \frac{1}{\varrho\lambda^2}\,\lambda W^2{}_\alpha = \frac{1}{\lambda^2\sigma^2}\left(V^0{}_\alpha + \frac{\sigma}{\varrho}\,V^2{}_\alpha\right) = \frac{1}{\lambda\sigma}\,W^0{}_\alpha\,,$$

and hence $\dot{z}_\lambda$ is constant. We can regard z_λ and $m\sigma\lambda W^0_\lambda = p_\lambda$ as conjugate co-ordinates and momenta.

The quantities W^1_λ appear as co-ordinates. We could regard them as internal co-ordinates for the particle (with z_α the external co-ordinates) in a classical bilocal theory corresponding to that introduced by YUKAWA [2]. Then eq. (5.4b) and (5.4c), which can be written as

(5.13)
$$\begin{cases} \dot{W}^1_\alpha = \dfrac{\lambda}{m}\dfrac{\partial H}{\partial W^2_\alpha}, \\[2mm] \dot{W}^2_\alpha = -\dfrac{\lambda}{m}\dfrac{\partial H}{\partial W^1_\alpha}, \end{cases}$$

can be regarded as the Hamiltonian equation for the internal motion and suggest that we regard W^2_λ as the momentum conjugate to the internal co-ordinates W^1_α.

It will be seen from the (P.b.) relations (5.11) that the (P.b.) relation between an « external variable » and an « internal variable » is zero.

6. – Concluding remarks.

The formalism developed in Section 5 suggests how the dynamics could be quantized.

Suppose that, in a quantum theory, the particle is described by a wave function $\psi = \psi(x_\alpha, W^\mu_\alpha)$ satisfying a differential equation

(6.1)
$$(H - h)\psi = 0,$$

where H is the operator corresponding to the Hamiltonian, and h an eigenvalue of H. The (P.b.) relations (5.11) suggest that the operator form for H is to be obtained by replacing W^0_λ and W^2_λ in the expression (5.5) for H, by operators $(-1/m\sigma\lambda)\partial/\partial z_\alpha$ and $(-\lambda/m)\partial/\partial W^1_\alpha$, respectively.

Because of (5.10) we can consider $\psi = \psi(z_\alpha, W^\mu_\lambda)$; and if we assume that a solution of (6.1) exists in the form

(6.2)
$$\psi = \varphi(z_\alpha)\theta(W^\mu_\alpha)$$

eq. (6.1) reduces to the pair of equations:

(6.3a)
$$\left(\frac{\partial^2}{\partial z_\alpha \partial z_\alpha} - \mu\right)\varphi = 0,$$

(6.3b)
$$\left(\frac{\partial^2}{\partial W^1_\alpha \partial W^1_\alpha} + W^1_\alpha W^1_\alpha + W^3_\alpha W^3_\alpha + \mu - h\right)\theta = 0.$$

[2] H. YUKAWA: *Phys. Rev.*, **77**, 219 (1950).

The former is an equation of the Klein-Gordon type but with the mass term μ determined by the solution of the eq. (6.3*b*) for the internal motion. This suggests that a mass spectrum will be obtained corresponding to various solutions of the equation for the internal motion. Also from (6.3*b*) there will appear extra quantum numbers in terms of which a possible particle classification could be obtained.

The quantization outlined above will be discussed in detail in a further paper.

The Nucleon-Nucleon Potential in Quantum Field Theory – I.

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Introduction.

The success met with by the ordinary Schrödinger equation in the study of nuclear phenomena at not too high energies supports the reasonable view that the interactions between nucleons may be described, at least within certain limits, by an ordinary potential. Besides this we have few pieces of information on the nuclear interactions and these essentially of a qualitative character only; we know that nuclear forces are short-range forces and have saturation properties; further we know that they depend on spin, on spin-orbit couplings, both static and velocity-dependent, and, perhaps somewhat less certainly, that they are not dependent on the charge state, once the exclusion principle is taken into account [1].

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[1] Books which may be consulted with profit in connection with the nuclear force problem are the following: J. M. Blatt and V. F. Weisskopf: *Theoretical Nuclear Physics* (New York, 1952); R. G. Sachs: *Nuclear Theory* (New York, 1953); A. S. Davydov: *Teoriya atomnogo yadra* (Moscow, 1958).

These qualitative facts together with general principles which we reasonably assume to hold in nuclear interactions [2], allow us to write a general expression for the nucleon-nucleon potential. (We shall restrict ourselves to this case, in the hope of losing no essential feature of the nuclear interaction.)

The general principles which we have mentioned are the following:

1) hermiticity $V = V^\dagger$;

2) translational and Galilean invariance which require, respectively, that the potential is a function of the relative co-ordinate r and the relative momentum p

$$V = V(\boldsymbol{\sigma}^{(1)}, \boldsymbol{\sigma}^{(2)}, \boldsymbol{r}, \boldsymbol{p}) ,$$

where

$$\boldsymbol{r} = \boldsymbol{r}^{(1)} - \boldsymbol{r}^{(2)} ,$$

$$\boldsymbol{p} = \tfrac{1}{2}(\boldsymbol{p}^{(1)} - \boldsymbol{p}^{(2)}) ,$$

$\boldsymbol{\sigma}^{(1)}$, $\boldsymbol{\sigma}^{(2)}$ are the spin operators for nucleon 1 and 2, respectively;

3) rotation and space reflection invariance, which requires the « scalar » nature of V;

4) particle exchange invariance, which requires

$$V(\boldsymbol{\sigma}^{(1)}, \boldsymbol{\sigma}^{(2)}, \boldsymbol{r}, \boldsymbol{p}) = V(\boldsymbol{\sigma}^{(2)}, \boldsymbol{\sigma}^{(1)}, -\boldsymbol{r}, -\boldsymbol{p}) ;$$

5) time reversal invariance which requires

$$V(\boldsymbol{\sigma}^{(1)}, \boldsymbol{\sigma}^{(2)}, \boldsymbol{r}, \boldsymbol{p}) = V^T(-\boldsymbol{\sigma}^{(1)}, -\boldsymbol{\sigma}^{(2)}, \boldsymbol{r}, -\boldsymbol{p}) ,$$

$$[(A_1 A_2 \dots A_{n-1} A_n)^T = A_n A_{n-1} \dots A_2 A_1] ;$$

6) charge-space rotation invariance (charge independence).

The quantities involved in the problem are the vectors $\boldsymbol{r}$, $\boldsymbol{p}$, $\boldsymbol{\sigma}^{(1)}$, $\boldsymbol{\sigma}^{(2)}$, $\underset{\rightarrow}{\tau}^{(1)}$, $\underset{\rightarrow}{\tau}^{(2)}$, where $\underset{\rightarrow}{\tau}^{(1)}$, $\underset{\rightarrow}{\tau}^{(2)}$ are the isotopic spin operators of nucleon 1 and 2, respectively [*].

Then the most general expression for the potential, if we allow a quadratic dependence on $\boldsymbol{p}$ at the most, depending on the tensor invariants which may be built up with the vectors above and satisfying the aforesaid condi-

[2] L. EISENBUD and E. P. WIGNER: *Proc. Nat. Acad. Sci. Wash.*, **27**, 281 (1941).
[*] We shall denote by heavy type the vectors of ordinary space and by a lower arrow the vectors of charge space.

tions, is [3]

$$V = V_0(r, p) + V_1(r, p)(\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)}) + V_2(r, p)S_{12} + V_3(r, p)(\boldsymbol{L} \cdot \boldsymbol{S}) +$$
$$+ V_4(r, p)\tfrac{1}{2}[(\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{L})(\boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{L}) + (\boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{L})(\boldsymbol{\sigma}^{(1)} \boldsymbol{L})] + V_5(r, p)(\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{p})(\boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{p}) +$$
$$+ \{V_6(r, p) + V_7(r, p)(\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)}) + V_8(r, p)S_{12} + V_9(r, p)(\boldsymbol{L}, \boldsymbol{S}) +$$
$$+ V_{10}(r, p)\tfrac{1}{2}[(\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{L})(\boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{L}) + (\boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{L})(\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{L})] + V_{11}(r, p)(\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{p})(\boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{p})\}(\boldsymbol{\tau}^{(1)} \cdot \boldsymbol{\tau}^{(2)}) ,$$

$$S_{12} = \frac{3(\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{r})(\boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{r})}{r^2} - (\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)}) , \qquad \boldsymbol{L} = \frac{1}{\hbar}(\boldsymbol{r} \times \boldsymbol{p}) , \qquad \boldsymbol{S} = \tfrac{1}{2}(\boldsymbol{\sigma}^{(1)} + \boldsymbol{\sigma}^{(2)}) .$$

The functions $V(r, p)$ are real functions of the moduli of $\boldsymbol{r}$, $\boldsymbol{p}$ and this is all that can be said of these functions on the basis of the aforesaid principles; the information of a general character which we mentioned at the beginning may give us certain qualitative indications (for instance they must practically vanish at a distance of the order 10^{-13} cm, the relative weight they must have in order to explain the data of the bound state and the scattering states, etc.) but no quantitative or detailed indication.

Three approaches are possible to the search of a form of these functions: a purely phenomenologic approach, a purely theoretical one and a mixed one. In the first case a potential is built up phenomenologically on the basis of the analysis of the experimental data: one example of a potential of this kind is the potential of Gammel, Christian and Thaler [4]; in the second case the problem of the two-nucleon interaction is treated according to the meson field theory, from which an expression for the potential between two nucleons is derived; in the third case a kind of fusion between the two approaches is done by taking what of « certain » is given by meson theories and by supplementing it by phenomenological information whenever the meson theories give unreliable results: one example of this kind is the Signell-Marshak potential [5].

In the past few years the two-nucleon problem has been the object of extensive study and widespread interest [6]; we think therefore that a critical account of the results reached up to now in the framework of meson field theories (*) might be of use [7].

[3] S. Okubo and R. E. Marshak: *Ann. Phys.*, **4**, 166 (1958).

[4] J. L. Gammel, R. S. Christian and R. M. Thaler: *Phys. Rev.*, **105**, 311 (1957).

[5] P. S. Signell and R. E. Marshak: *Phys. Rev.* **106**, 832 (1957); **109**, 1229 (1958).

[6] A detailed survey of the phenomenological aspect of the problem may be found in the article by L. Hulthén and M. Sugawara in: *Handbuch der Physik*, vol. XXXIX (Berlin, 1957). More recent results can be found in: *Nuclear Forces and the Few-Nucleon Problem*, vol. I (London, 1960).

(*) Within that particular approach called « weak coupling approach ».

[7] Review articles of wider subjects in which this problem is treated are: R. J. N. Phillips: *Rep. Prog. Phys.*, **22** (1959); R. H. Dalitz: *Prog. Nucl. Phys.*, **4** (1955).

1. – Conditions and limits of a theoretical potential; formulation of the problem.

1˙1. – Before formulating the problem of the construction of the potential, in the framework of meson theories, it is expedient to point out certain conditions which must be met by the potential and which are essential to the concept of potential itself as a description, valid only within certain limits, of the complete interaction between the two-nucleons ([8-10]):

1) it must be hermitian;

2) it must be defined only for energies of the nucleons lower than the threshold for meson production;

3) it must be independent of the total energy W of the system;

4) it must be such that the solutions of the Schrödinger equation give for the binding energy of the two bound nucleons and for the phase shifts in the scattering processes the same results as those derived directly from field theory ([*]).

These conditions impose on the potential operator restrictions both in the r dependence and the p dependence.

The condition 2) may be written, in the center-of-mass system,

$$\frac{p^2}{M} < \mu_\pi c^2$$

(M is the nucleon mass and μ_π is the π-meson mass).

Considering p as the nucleon recoil momentum and applying the uncertainty relation, we obtain that a theoretical definition for the potential shall be possible only for nucleon separations $r > r_{\min}$ with $r_{\min} \approx 0.55 \cdot 10^{-13}$ cm, i.e. for a separation larger than about one third of the meson Compton wavelength. (We shall see later other reasons supporting this limitation.)

The region $r \leqslant r_{\min}$ therefore must be excluded from a theoretical treatment and, if one wishes to retain the description of the interaction within this region

([8]) A. KLEIN and B. H. MC CORMICK: Phys. Rev., 104, 1747 (1956).

([9]) K. NISHIJIMA: Suppl. Progr. Theor. Phys., 3, 138 (1956).

([10]) M. TAKETANI, S. NAKAMURA and M. SASAKI: Progr. Theor. Phys., 6, 581 (1951) (reprinted in Suppl. Progr. Theor. Phys., 3, 169 (1956).

([*]) Phase shifts have in the framework of field theory a well-defined meaning. Bound states may be introduced in the following way. The variety of the solutions of the field equation contains a subvariety of solutions corresponding to a value of the mass of the system of two particles which is less than the sum of the masses of the two particles: it is assumed that this subvariety describes bound states.

by means of a potential, this can be done only on a phenomenological basis. For this purpose the assumption of a hard core is at present the most common attitude.

As for the p dependence, the conception of a theoretical potential, as has been outlined above, is valid and useful for small relative nucleon momenta. This implies that the retardation effects in the transmission of the interactions between the two nucleons are small, thereby suggesting the inclusion of quadratic terms in p at most.

1'2. – In the framework of field theory the problem may be formulated in the following way. The field equation

$$(1.1) \qquad i\hbar \frac{\partial}{\partial t} |\Phi(t)\rangle = H |\Phi(t)\rangle ,$$

or the corresponding stationary state equation

$$(1.1') \qquad (H - W)|\Phi\rangle = 0$$

must be reduced, in an appropriate variety of state vectors (see note p. 140), to a Schrödinger equation which should, within certain limits, be physically equivalent to each of them respectively:

$$(1.2) \qquad i\hbar \frac{\partial}{\partial t} |\varphi(t)\rangle = \left(\frac{p^2}{M} + V\right)|\varphi(t)\rangle ,$$

$$(1.2') \qquad \left(\frac{p^2}{M} + V\right)|\varphi\rangle = W |\varphi .$$

Here the ket $|\Phi\rangle$ is the field functional, H the total field Hamiltonian, sum of the free field Hamiltonian H_0 ($H_0 = H_N + H_M$) and of the interaction Hamiltonian H'

$$H = H_0 + H = H_N + H_M + H' ,$$

W is the total energy of the system and the ket $|q$ is the ordinary Schrödinger ket describing the state of two physical nucleons.

1'3. – The question may be posed if a potential meeting all the requirements stated above can actually be found. The problem may be conveniently discussed in terms of S-matrix quantities and, as it will be seen from the following discussion, the answer will turn out to be, within certain limits, positive [11].

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[11] M. A. Braun: *Soviet Phys. J.E.T.P.*, **10**, 582 (1960).

Let us treat the two-nucleon system in ordinary wave mechanics and let us denote by $|\varphi_\lambda\rangle$ the eigenvectors of the unperturbed Hamiltonian h_0, by $|\psi_\alpha^{(+)}\rangle$ the eigenvectors of the total Hamiltonian $h_0 + V$ with outgoing waves, by $|\psi_m\rangle$ the eigenvectors of the total Hamiltonian corresponding to bound states (it will be $W_m < W_\alpha$ when the total momenta of these states are the same).

An element of the t-matrix is defined by the relation

$$(1.3) \qquad\qquad t_{\alpha\beta} = \langle \varphi_\lambda | V | \psi_\beta^{(+)}\rangle$$

from which it follows that

$$\langle \varphi_\lambda | \psi_\beta^{(+)}\rangle = \delta_{\lambda\beta} - t_{\alpha\beta}/(E_\alpha - E_\beta - i\eta) \, .$$

The link with the s-matrix is given by the relation

$$(1.4) \qquad\qquad s_{\lambda\beta} = \delta_{\alpha\beta} - 2\pi i \, \delta(E_\lambda - E_\beta) t_{\alpha\beta} \, .$$

Matrix elements $t_{\alpha m}$ may be defined in an analogous way by the relation

$$\langle \varphi_\alpha | \psi_m\rangle = t_{\alpha m}/(E_m - E_\alpha) \, .$$

The matrix elements $t_{\lambda\beta}$ are connected to the matrix elements of the potential operator by the well-known Lippmann-Schwinger equation:

$$(1.5) \qquad\qquad t_{\alpha\beta} = V_{\alpha\beta} - \sum_\gamma \frac{V_{\alpha\gamma} t_{\gamma\beta}}{E_\gamma - E_\beta - i\eta} \, .$$

This may be solved with respect to the potential, making use of the completeness of the set $|\psi_\beta^{(+)}\rangle$, $|\psi_m\rangle$ [12]:

$$(1.6) \qquad\qquad V_{\lambda\beta} = t_{\alpha\beta} + \sum_\gamma \frac{t_{\alpha\gamma} t_{\gamma\beta}^\dagger}{E_\gamma - E_\beta - i\eta} + \sum_m \frac{t_{\alpha m} t_{m\beta}^\dagger}{E_m - E_\beta} \, .$$

The conditions of orthonormality and completeness of the sets $|\psi_\beta^{(+)}\rangle$, $|\psi_m\rangle$ on one side and $|\varphi_\beta\rangle$ on the other impose on the $t_{\gamma\beta}$ and $t_{\alpha m}$ certain restrictions which are typified by the following equation

$$(1.7) \qquad \frac{t_{\alpha\beta}^\dagger - t_{\alpha\beta}}{E_\alpha - E_\beta - i\eta} + \sum_\gamma \frac{t_{\alpha\gamma} t_{\gamma\beta}^\dagger}{(E_\alpha - E_\gamma - i\eta)(E_\beta - E_\gamma + i\eta)} + $$

$$+ \sum_m \frac{t_{\alpha m} t_{m\beta}^\dagger}{(E_m - E_\alpha)(E_m - E_\beta)} = 0 \, .$$

[12] Yu. V. Novozhilov: Soviet Phys. J.E.T.P., 8, 515 (1959).

where the summations are over the spins and momenta of the two particles and their bound states.

These conditions are very important and assure that (1.5) with the potential V will have as its solution the same $t_{\alpha\beta}$ as appear in (1.6), and that V is Hermitian.

In quantum field theory the elements of the T-matrix are defined by a relation analogous to (1.3):

$$(1.8) \qquad T_{\Gamma A} = \langle \Phi_\Gamma | H - E_\Gamma | \Psi_A^{(+)} \rangle \, ,$$

where $|\Phi_\Gamma\rangle$ are asymptotically stationary states, which are the analogous of the states of non-interacting particles in ordinary wave mechanics (*), $|\Psi_1^{(+)}\rangle$ eigenstates with outgoing waves, which develop from the asymptotically stationary states $|\Phi_A\rangle$ with energy E_A, H is the total field Hamiltonian. The S-matrix is expressed in terms of the $T_{\Gamma A}$ by a formula analogous to (1.4). Beside the $T_{\Gamma A}$ we may define the analogous matrix elements

$$T_{\Gamma M} = \langle \Phi_\Gamma | H - E_\Gamma | \Psi_M \rangle \, ,$$

where $|\Psi_M\rangle$ are bound states in field theory.

It might be thought to assume the field-theoretical $T_{\alpha\beta}$ and $T_{\alpha m}$ as wave-mechanical transition amplitudes $t_{\nu\beta}$ and $t_{\alpha m}$ to be inserted into eqs. (1.3) to (1.6); the potential would then be defined by (1.6).

This identification is prevented by the following argument.

The quantities $t_{\alpha\beta}$ and $t_{\alpha m}$ which one inserts in (1.6) must satisfy conditions of the type (1.7); now the quantities $T_{\alpha\beta}$ and $T_{\alpha m}$ do satisfy equations of the type (1.7) but with summation taken over all the field-theoretical states, including therefore, as well as states with $\Gamma = \gamma$ and $M = m$, (corresponding to the two particles we are treating), also states with $\Gamma \neq \gamma$ and $M \neq m$ (corresponding to other sets of particles).

The relations satisfied by the $T_{\lambda\beta}$ and $T_{\lambda m}$ are therefore different from those prescribed for the $t_{\nu\beta}$ and $t_{\nu m}$.

The attempted identification may be reached for other quantities $\widetilde{T}_{\alpha\beta}$, $\widetilde{T}_{\alpha m}$ which we will define presently. Let us separate from all the states $|\Psi^{(+)}\rangle$ the states with $\Gamma = \gamma$: let P be the operator for projection onto these states, so that

$$(1.9) \qquad \langle \Psi' | P \Psi'' \rangle = \sum_\gamma \langle \Psi' \Psi_\gamma^{(+)} \rangle \langle \Psi_\gamma^{(+)} | \Psi'' \rangle + \sum_m \langle \Psi' | \Psi_m \rangle \langle \Psi_m | \Psi'' \rangle \, .$$

(*) These asymptotically stationary states may be thought of as those introduced by VAN HOVE ([13]) or by NOVOZHILOV ([12]); in most common S-matrix considerations they are often assumed to be eigenstates of the free Hamiltonian H_0.

([13]) L. VAN HOVE: *Physica*, **21**, 901 (1955); **22**, 343 (1956).

We define the state vectors

$$(1.10) \qquad |\widetilde{\Phi}_\alpha\rangle = \sum_\gamma [1 + N]_{\gamma\alpha}^{-\frac{1}{2}} |\Phi_\gamma\rangle \,,$$

where N is given by

$$(1.11) \qquad \langle\Phi_\alpha | P\Phi_\beta\rangle = \delta_{\alpha\beta} + N_{\alpha\beta} \,.$$

The states $|P\widetilde{\Phi}_\alpha\rangle$ are orthonormalized

$$\langle\widetilde{\Phi}_\gamma | P\widetilde{\Phi}_\beta\rangle = \delta_{\alpha\beta} \,,$$

and are assumed to be a complete set in the subspace determined by the projection operator P.

If we define matrix elements $\widetilde{T}_{\alpha\beta}$ and $\widetilde{T}_{\gamma m}$ through the following relations

$$(1.12) \qquad \langle\widetilde{\Phi}_\alpha | \Psi_\beta^{(+)}\rangle = \delta_{\alpha\beta} - \frac{\widetilde{T}_{\alpha\beta}}{E_\alpha - E_\beta - i\eta} \,,$$

$$(1.13) \qquad \langle\widetilde{\Phi}_\alpha | \Psi_m\rangle = \frac{\widetilde{T}_{\alpha m}}{E_m - E_\alpha} \,,$$

the $\widetilde{T}_{\alpha\beta}$ and $\widetilde{T}_{\gamma m}$ satisfy, in virtue of the orthonormality and completeness of the states $|\Psi_\gamma^{(+)}\rangle$, $|\Psi_m\rangle$ on one hand and of the $|P\widetilde{\Phi}_\gamma\rangle$ on the other, the prescribed conditions (1.7). Then, as far as conditions (1.7) are concerned, nothing prevents us from inserting the $\widetilde{T}_{\gamma\beta}$ in place of the $t_{\gamma\beta}$ into the definition of the potential.

All this is, obviously, up to now purely formal; but it can be shown that the $\widetilde{T}_{\gamma\beta}$ are identical with the $T_{\gamma\beta}$ on the energy shell ($E_\gamma = E_\beta$), so far as processes of creation and annihilation are impossible.

Indeed from (1.11) we have

$$N_{\gamma\beta} = \langle\Phi_\gamma | \Phi_\beta\rangle - \delta_{\gamma\beta} - \sum_{\Gamma \neq \gamma} \langle\Phi_\gamma | \Psi_\Gamma^{(+)}\rangle\langle\Psi_\Gamma^{(+)} | \Phi_\beta\rangle \,.$$

This expression has no singularities for $E_\gamma = E_\beta$ if processes of creation and annihilation do not occur; therefore, as can be seen from (1.10), the singularities of $\langle\widetilde{\Phi}_\alpha | \Psi_\gamma^{(+)}\rangle$ and $\langle\Phi_\alpha | \Psi_\gamma^{(+)}\rangle$ are the same under these circumstances.

Then

$$\lim_{E_\chi \to E_\beta} (E_\alpha - E_\beta)\langle\widetilde{\Phi}_\alpha | \Psi_\beta^{(+)}\rangle = \lim_{E_\chi \to E_\beta} (E_\alpha - E_\beta)\langle\Phi_\alpha | \Psi_\beta^{(+)}\rangle \,,$$

which indeed prooves the assertion we have made.

The « potential » is so defined by the following relation:

$$(1.14) \quad V_{\alpha\beta} = \widetilde{T}_{\alpha\beta} + \sum_{\gamma} \frac{\widetilde{T}_{\alpha\gamma}\widetilde{T}^{\dagger}_{\gamma\beta}}{E_{\gamma} - E_{\beta} - i\eta} + \sum_{m} \frac{\widetilde{T}_{\alpha m}\widetilde{T}^{\dagger}_{m\beta}}{E_{m} - E_{\beta}} =$$

$$= \langle \widetilde{\Phi}_{\alpha} | H - E_{\alpha} | P\widetilde{\Phi}_{\beta} \rangle + \sum_{\gamma} \langle \widetilde{\Phi}_{\alpha} | H - E_{\alpha} | \Psi^{(+)}_{\gamma} \rangle \langle \Psi^{(+)}_{\gamma} | \Psi^{(+)}_{\beta} - P\widetilde{\Phi}_{\beta} \rangle +$$

$$+ \sum_{m} \langle \widetilde{\Phi}_{\alpha} | H - E_{\alpha} | \Psi_{m} \rangle \langle \Psi_{m} | \Psi^{(+)}_{\beta} - P\widetilde{\Phi}_{\beta} \rangle + \sum_{\gamma} \frac{\widetilde{T}_{\alpha\gamma}\widetilde{T}^{\dagger}_{\gamma\beta}}{E_{\gamma} - E_{\beta} - i\eta} \div$$

$$+ \sum_{m} \frac{\widetilde{T}_{\alpha m}\widetilde{T}^{\dagger}_{m\beta}}{E_{m} - E_{\beta}} = \langle \Phi_{\alpha} | H - E_{\alpha} | P\widetilde{\Phi}_{\beta} \rangle \ .$$

We see so that, for energies lower than the creation threshold, it is possible to define, in the framework of field theory, a « potential » which correctly predicts the field theoretical cross-sections.

However, two important remarks must be added to the discussion outlined: the first regards the uniqueness of the definition, the second the non-local character of the « potential ». The « potential » defined above is not unique: the discussion is unchanged if in place of the set $\widetilde{\Phi}_{\gamma}$ another set is used:

$$(1.10') \qquad\qquad \widetilde{\Phi}'_{\alpha} = \sum_{\beta} U_{\beta\alpha}\widetilde{\Phi}_{\beta} \ ,$$

where U is an arbitrary unitary operator, which can be written in the form $U = 1 + U_{1}$, with U_{1} a non-diagonal matrix which goes to zero when $N = 0$ and which has no singularities at $E_{\gamma} = E_{\beta}$ when $N_{\gamma\beta}$ has no such singularities; in this way one obtains a « potential »

$$(1.14') \qquad\qquad V'_{\alpha\beta} = \sum_{\gamma_1 \gamma_2} U^{\dagger}_{\alpha\gamma_1} \langle \widetilde{\Phi}_{\gamma_1} | H - E_{\alpha} | P\widetilde{\Phi}_{\gamma_2} \rangle U_{\gamma_2\beta} \ ,$$

which is *not* a unitary transformation of V and which may describe scattering processes and bound states equally well as the original V.

As for the second feature it is easy to see that the « potential » which we have defined is a non-local one; the equation equivalent to (1.1') reads

$$(1.15) \qquad\qquad (2E(\boldsymbol{p}) - W)\, \varphi(\boldsymbol{p}) = \frac{1}{\hbar^3} \int \mathscr{V}(\boldsymbol{p}, \boldsymbol{p}')\, \varphi(\boldsymbol{p})' \, \mathrm{d}\boldsymbol{p}' \ ,$$

with

$$E(\boldsymbol{p}) = c\sqrt{c^2 M^2 + \boldsymbol{p}^2} \ .$$

Notice that the operator $\mathscr{V}(\boldsymbol{p}, \boldsymbol{p}')$ depends on the momenta not through $\boldsymbol{p} - \boldsymbol{p}'$ but through a more general dependence; accordingly its Fourier trans-

form in co-ordinate space is of the form $\mathscr{V}(\boldsymbol{r}^{(1)} - \boldsymbol{r}^{(2)}, \boldsymbol{r}^{(1)'} - \boldsymbol{r}^{(2)'})$ and is therefore a non-local operator.

Strictly speaking a local potential cannot be defined, if it is to describe the scattering correctly. The scattering amplitude, as a function of energy is, for a local potential, analytic in the upper half-plane and on the real axis. If it is identical with the field theoretical amplitude, which is also an analytic function in the same region, on a segment of the real axis, it is also identical, on account of analiticity, on the whole real axis.

But this cannot be true, since above the threshold energy, creation and annihilation processes do occur, and these, clearly, cannot be accounted for by the wave-mechanical amplitude.

In simple terms we may say that, even under the energy threshold the scattering amplitude « feels » the presence of singularities due to creation and annihilation processes; at an energy under the threshold far enough from these singularities, however, it may be expected that their effect on the scattering amplitude is small and the scattering may be described with a good approximation by a local potential. That this statement is plausible can be seen from the treatment of spinless particles by CHARAP and FUBINI (^{14}) who have shown that, for this case, the field-theoretical amplitude and the amplitude derived from a local potential are the same under the threshold when neglecting terms which are $O(p^2/p_{\text{thr}}^2)$ (*).

In view of this fact the possibility of attaining the construction of a potential with a local character in quantum field theory, for separations larger than about one third of the pion Compton wavelength, appears somewhat impaired; it seems likely that it will be possible to do so only for larger separations.

In general the potential V is defined as the limit of the operator $\mathscr{V}$ when $p/Mc \ll 1$ (this is called the « adiabatic limit »).

For this discussion we have used concepts and quantities belonging to the S-matrix formalism; other methods for the construction of the potential, as it will be seen, may be used, still in the framework of field theory, which yield, as an equation equivalent to (1.1'), an equation of the form

$$(1.16) \qquad (2E(\boldsymbol{p}) - W)\,\varphi(\boldsymbol{p}) = \frac{1}{\hbar^3} \int \mathscr{V}(\boldsymbol{p}, \boldsymbol{p}';\, W)\,\varphi(\boldsymbol{p}')\,\mathrm{d}\boldsymbol{p}' \;,$$

where the operator $\mathscr{V}$ explicitly depends on W, the total energy of the system.

(^{14}) J. M. CHARAP and S. P. FUBINI: *Nuovo Cimento*, **14**, 540 (1959); **15**, 73 (1960); see also M. CINI and S. FUBINI: *Ann. Phys.*, **10**, 352 (1960).

(*) A similar proof for particles with spin, independent of perturbation theory, has not been possible up to now because the dispersion relations, on which the treatment of Charap and Fubini is based, have not yet been proved for this case.

Several ways have been devised to eliminate this unwanted dependence, all of which, however, involve a large amount of ambiguity,

All these methods, which fully employ quantum field theory, are called « non-adiabatic methods », in contrast to the old-fashioned methods, which were called « adiabatic methods ».

These methods assumed the period of the nucleon motion to be large relative to the time needed to exchange mesons; the two nucleons were then regarded as approximately at rest during the exchange of mesons; the problem with fixed nucleons was then solved as a first step and the eigenvalue thus obtained, which depended parametrically on r, was adopted as a potential for the problem of the nucleon motion. In more precise terms the idea underlying these methods was the limit $\mu_\pi/M \to 0$ (this is called the « static limit »). Now this limit is in general not justified, as CHARAP and FUBINI have shown and therefore the adiabatic methods (like any fixed source approach) can only have the meaning of a model approach, not of a limit of a complete field theory.

1˙4. – The basic attitude regarding the eq. (1.1′) is well known: the existence of an analytic solution is assumed and this is built up by means of some series expansion. This is the typical attitude underlying the so-called « weak coupling approach ».

The potential shall therefore have the form of a series expansion. It is clear (as comparison with electrodynamics shows) that an expansion in powers of the mathematical coupling constant is meaningless since the terms of the expansion exhibit the well-known divergences. On the other hand an expansion in the number of exchanged mesons may be significant. It is hoped that such an expansion, at least in the adiabatic limit, is equivalent to an expansion in powers of the renormalized coupling constant and with all other quantities properly renormalized; then, apart from questions of convergence, one has to do with a « clean » series. It must be pointed out that in a series expansion of this type the problem of convergence appears, in the case of the nuclear potential, in less critical terms than in other applications of field theory. Wick's classic argument ([15]) in fact shows that the terms due to exchange of 3, 4, ... mesons will be important at separations between the two nucleons less than about $h/3\mu_\pi c$, $h/4\mu_\pi c$, ..., i.e. in a region which, as we have seen, must be excluded from our theoretical consideration. It is therefore reasonable to think that the first two terms in the expansion in the number of exchanged mesons give an adequate description of the interaction between the two nucleons in the region where the interaction may be described by a theoretical potential.

([15]) G. C. WICK: *Nature*, **142**, 993 (1938).

1`5. – In order to write the total field Hamiltonian we must take into account on one hand the already stated general principles (see 1`1), except that the principle of invariance under Lorentz transformations must be substituted for the principle of Galilean invariance; on the other hand we must take into account the available phenomenological information. The range of nuclear forces and our qualitative knowledge on the interaction of mesons with nucleons compel us to exclude μ mesons and to regard π mesons as those mainly responsible for nuclear forces. As for the heavy mesons, which we collectively denote as K mesons, it seems likely that they play an important role in nuclear forces, although their part is not yet quantitatively known. From the uncertainty relation it is easy to see that if K mesons play a part in nuclear forces they do so to an appreciable amount within a radius of the order of $r_K = \hbar/\mu_K c \approx \frac{1}{3}\hbar/\mu_\pi c$.

Meson theories have so far only taken into account π mesons; it is obvious that such theories can claim no validity for $r < r_K$ (*).

Confining the consideration to π mesons (**), two other essential properties are given by experiment; spin (which is equal to 0) and intrinsic parity (which is negative). These bring us to regard the meson field as a pseudoscalar field. Finally, if we add the assumption that the interaction is linear in the meson field variables and in their first derivatives, we can use two types of interaction Hamiltonian

$$(1.17) \qquad H'_{ps} = ig_1 \int \overline{\psi}\underset{\rightarrow}{\tau}\gamma_5\psi\underset{\rightarrow}{\Phi}\,d\boldsymbol{x}, \qquad\qquad \text{(ps-ps theory)},$$

$$(1.18) \qquad H'_{pv} = \frac{ig_2}{\varkappa} \int \overline{\psi}\underset{\rightarrow}{\tau}\gamma_5\gamma_\mu\psi\,\frac{\partial\underset{\rightarrow}{\Phi}}{\partial x_\mu}\,d\boldsymbol{x}, \qquad \text{(ps-pv theory) (**)},$$

where ψ, $\overline{\psi} = \psi^\dagger\gamma_4$, are the nucleon field operators, $\underset{\rightarrow}{\Phi}$ (a vector in charge space) is the meson field operator, $\underset{\rightarrow}{\tau}$ is the isotopic spin operator for the nucleon, $\varkappa = \mu c/\hbar$, g_1 and g_2 are the coupling constant for the pseudoscalar and the pseudovector coupling, respectively, $\gamma_r = -i\beta\alpha_r$ ($r = 1, 2, 3$), $\gamma_4 = \beta$, $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$ (α_r and β are the usual Dirac matrices).

The Hamiltonians (1.17) and (1.18) are in accordance with the so-called symmetric theory of Kemmer ([16]), which ensures charge independence. Another

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(*) This is another reason for limiting the validity of the description of the interaction between the two nucleons by a potential derived from pion theories to separations $r \geqslant \frac{1}{3}(\hbar/\mu_\pi c)$.

(**) Accordingly we shall hereafter write μ without index for the pion mass.

(**) A contact term may be omitted since it contributes a term of zero range to the two-nucleon interaction.

([16]) N. Kemmer: *Proc. Cambridge Phil. Soc.*, **34**, 354 (1938).

way to ensure charge independence is to use the neutral pseudoscalar theory whose interaction Hamiltonians are

$$(1.19) \qquad H'_{\mathrm{ps}} = ig_1 \int \overline{\psi}\gamma_5\psi\Phi\,\mathrm{d}\boldsymbol{x}\,,$$

$$(1.20) \qquad H'_{\mathrm{pv}} = i\frac{g_2}{\varkappa}\int \overline{\psi}\gamma_5\gamma_\mu\psi\,\frac{\partial\Phi}{\partial x_\mu}\,\mathrm{d}\boldsymbol{x}\,,$$

where Φ is a scalar in charge space.

Certains terms of the potential built up from (1.17), (1.18) contain the factor $(\underset{\rightarrow}{\tau}^{(1)}\cdot\underset{\rightarrow}{\tau}^{(2)})$; if one substitutes this factor by unity one obtains the potential yielded by (1.19) and (1.20); we shall therefore always use (1.17) and (1.18).

We also quote the form taken by the Hamiltonian when assuming the source as fixed (« fixed source theory ») which has been used by many authors for the calculation of the potential $(K = Mc/\hbar)$:

$$(1.21) \qquad H'_{\mathrm{ps}} = \frac{g_1}{2K}\int\varrho(\boldsymbol{x})(\boldsymbol{\sigma}\cdot\nabla)(\underset{\rightarrow}{\tau}\cdot\underset{\rightarrow}{\Phi})\,\mathrm{d}\boldsymbol{x} + \frac{g_1^2}{2K\hbar c}\int\varrho(\boldsymbol{x})(\underset{\rightarrow}{\tau}\cdot\underset{\rightarrow}{\Phi})^2\,\mathrm{d}\boldsymbol{x}\,,$$

$$(1.22) \qquad H'_{\mathrm{pv}} = \frac{g^2}{\varkappa}\int\varrho(\boldsymbol{x})(\boldsymbol{\sigma}\cdot\nabla)(\underset{\rightarrow}{\tau}\cdot\underset{\rightarrow}{\Phi})\,\mathrm{d}\boldsymbol{x}\,,$$

where $\varrho(\boldsymbol{x})$ is the fixed source density, normalized in the usual way $\int\varrho(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}=2$; if, as frequently done, point sources are assumed, $\varrho(\boldsymbol{x})$ becomes

$$\varrho(\boldsymbol{x}) = \sum_{i=1}^{2}\delta(\boldsymbol{x} - \boldsymbol{r}^{(i)})\,.$$

It will be noticed that the forms (1.21) and (1.22) are not invariant under Galilean transformations.

It is easy to see the physical meaning of the two terms of (1.21). The first term describes the interaction between a fixed nucleon and a p-wave meson.

The second term has been the subject of debate ([17-23]) a few years ago. It has its origin from processes of virtual nucleon pair creation and annihilation

[17] S. D. DRELL and E. M. HENLEY: Phys. Rev., 88, 1053 (1952).

[18] G. WENTZEL: Phys. Rev., 86, 802 (1953).

[19] K. A. BRUECKER, M. GELL-MANN and M. L. GOLDBERGER: Phys. Rev., 90, 476 (1953).

[20] N. M. KROLL and M. A. RUDERMAN: Phys. Rev., 93, 233 (1954).

[21] S. DESER, W. E. THIRRING and M. L. GOLDBERGER: Phys. Rev., 94, 711 (1954).

[22] A. KLEIN: Phys. Rev., 95, 1061 (1954).

[23] S. F. EDWARDS and P. T. MATTHEWS: Phil. Mag., 2, 176, 467 (1957).

and would seem to dominate the interaction. Arguments have been brought forward, however, which support the belief that the effect of these processes is largely overvalued by the second term of (1.21) as compared with their treatment by means of the complete Hamiltonian, at least at not too high energies. In other words these processes must be strongly damped (perhaps by a factor 10), if the description of the interaction by means of the Hamiltonian (1.21) is to be retained. The arguments in favour of a strong damping are of a qualitative rather than a quantitave character and, although supporting the belief that these precesses are not of decisive effect in the mechanism of nuclear forces, do not supply a definite prescription for the evaluation of their role. The Hamiltonian (1.21) is therefore modified by the inclusion in the second term of a factor λ, presumably very small:

$$(1.23) \qquad H'_{\text{ps}} = \frac{g_1}{2K} \int \varrho(\boldsymbol{x})(\boldsymbol{\sigma} \cdot \nabla)(\underset{\rightarrow}{\tau} \cdot \underset{\rightarrow}{\varPhi}) \, \mathrm{d}\boldsymbol{x} + \lambda \frac{g_1^2}{2Khc} \int \varrho(\boldsymbol{x})(\underset{\rightarrow}{\tau} \cdot \underset{\rightarrow}{\varPhi})^2 \, \mathrm{d}\boldsymbol{x} \,.$$

The contributions to the potential given by the second term will be easily identified by the factor λ.

In the case $\lambda = 0$, i.e. in case the second term of (1.24) is negligible, the Hamiltonian (1.21) becomes the Hamiltonian (1.22), provided $g_1/2K = g_2/\varkappa$.

This means that up to a certain order of approximation there are processes which are described in an equivalent way both by the ps coupling and by the pv coupling; this property was known as « Dyson's equivalence theorem » ([24]).

Of these « fixed source theories » we must however remind what we said when discussing the static limit ($\mu/M \to 0$).

2. – Methods for the construction of the potential.

In the past the methods more frequently employed have been the adiabatic methods ([25]). According to these methods the potential was defined as the lowest eigenvalue of the Hamiltonian

$$H_{\text{ad}} = H_M + H'_{\text{ad}} \,,$$

where H_M is the free meson field Hamiltonian and H'_{ad} is the fixed source

([24]) F. J. DYSON: Phys. Rev. **73**, 929 (1948).

([25]) N. KEMMER: Proc. Roy Soc. of London, A **166**, 127 (1938); H. FRÖHLICH, W. HEITLER and N. KEMMER: Proc. Roy. Soc. (London), A **166**, 154 (1938); C. MØLLER and L. ROSENFELD: Kgl. Dan. Vid. Selskab., **17**, 8 (1946); G. WENTZEL: Quantum theory of fields (New York, 1949), p. 63; J. IWADARE: Progr. Theor. Phys., **13**, 189 (1955); **14**, 16 (1955); C. MØLLER: Pseudoscalar meson theory, CERN, 55-19 (1955).

interaction Hamiltonian (given by (1.21) or (1.22)). The calculation of this eigenvalue was attained either by means of ordinary perturbation theory or by means of canonical transformations. Since the static limit was inherent to these methods they are now felt to be inadequate and obsolete.

There are various non-adiabatic methods, which may be classified according to four approaches: a) Tamm-Dancoff method; b) S-matrix methods; c) method of Brueckner and Watson; d) method of the Bethe-Salpeter equation.

2'1. *Tamm-Dancoff method* [26,27]. – In the Tamm-Dancoff method the equation (1.1′) is transformed into an infinite set of simultaneous integral equations for the amplitudes $a_m^n(\lambda)$ of the free Hamiltonian eigenstates (where m is the meson number, n the number of nucleon-antinucleon pairs, λ refers to the momenta, spins and isotopic spins).

The eigenvectors $|m, n, \lambda\rangle$ of the free Hamiltonian are defined by the equation

(2.1) $$H_0|m, n, \lambda\rangle = E_m^n(\lambda)|m, n, \lambda\rangle$$

and form a complete orthonormal set. The state functional $|\Phi\rangle$ may then be expanded into a series of the $|m, n, \lambda\rangle$:

(2.2) $$|\Phi\rangle = \sum_{m,n,\lambda} a_m^n(\lambda)|m, n, \lambda\rangle .$$

Substituting this expansions into the equation (1.1′) one obtains

(2.3) $$\sum_{m,n,\lambda} a_m^n(\lambda) E_m^n(\lambda)|m, n, \lambda\rangle + \sum_{m,n,\lambda} a_m^n(\lambda) H'|m, n, \lambda\rangle = W \sum_{m,n,\lambda} a_m^n(\lambda)|m, n, \lambda\rangle$$

and from this the following set of integral equations:

(2.4) $$[W - E_m^n(\lambda)]a_m^n(\lambda) = \sum_{p,q,\mu} \langle m, n, \lambda|H'|p, q, \mu\rangle a_p^q(\mu) .$$

If the interaction H' is assumed to be the ps-interaction (*)

$$H' = ig_1 \int \overline{\psi}\underset{\rightarrow}{\tau}\gamma_5\psi\underset{\rightarrow}{\Phi}\,\mathrm{d}\boldsymbol{x} ,$$

[26] I. TAMM: *Žurn. Èksp. Teor. Fiz.*, **9**, 449 (1945).
[27] S. M. DANCOFF: *Phys. Rev.*, **78**, 382 (1950).
(*) The ps coupling has the advantage that it allows the renormalization to be carried out with Dyson's techniques, what cannot be done for the pv coupling theory.

this set takes the following form:

$$(2.5) \qquad W - E^n_m(\lambda)]a^n_m(\lambda) = \sum_{p=m\pm1} \sum_{q=n-1}^{n+1} \sum_{\mu} \langle m, n, \lambda \,|\, H' \,|\, p, q, \mu \rangle a^q_p(\mu) \,.$$

The set (2.5) is a mathematically exact formulation of the problem, but it involves an infinite number of equations. An approximate way to solve it is that introduced by TAMM and DANCOFF. The approximation introduced lies in putting equal to zero the probability amplitudes a^n_m for $n > N$ and $m > M$ (with M, N fixed numbers). A set of a finite number of equations is thus obtained which may, in principle, be solved in an exact way.

For the consistency of the method it is to be hoped that the solutions converge for N, $M \to \infty$; in other words that the results of the calculation are insensitive to the number of particles N, M when N, M are large enough and tend to a finite limit when N, $M \to \infty$. Unfortunately this is only a hope because nobody has ever solved the set of equations but for the case of very few components and the assumption therefore lacks any support.

A second method to solve the set (2.5) is that used by LÉVY ([28]). This consists in the successive algebraic elimination by induction of the amplitudes of the states containing mesons and nucleon-antinucleon pairs. What remains after this elimination is an integral equation for the amplitude of the zero-meson zero-pair state, in which the kernel has the form of a power series expansion in the coupling constant. In the actual solution the kernel is then cut at a certain order of the coupling constant; this implies the assumption of the convergence of this expansion. The equation thus obtained has the following form:

$$(2.6) \qquad [W - E^0_0(\lambda)]a^0_0(\lambda) = \sum_{\mu} K(\lambda, \mu; W)a^0_0(\mu) \,.$$

The kernel K as the form

$$(2.7) \qquad K = \sum_q K_{2q}(\lambda, \mu; W) \,,$$

with

$$(2.8) \qquad K_{2q}(\lambda, \mu; W) \propto g^{2q}_1 \,.$$

Eq. (6) appears like a Schrödinger equation; there are reasons however which prevent it to be interpreted as such. In fact the a^0_0 appearing in the equation cannot satisfy the normalization condition, since this for the system in ques-

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([28]) M. M. LÉVY: *Phys. Rev.*, **88**, 72, 725 (1952).

tion is

$$(2.9) \qquad \sum_{m,n,\lambda} |a_m^n(\lambda)|^2 = 1 \,,$$

and the condition

$$(2.10) \qquad \sum_{\lambda} |a_0^0(\lambda)|^2 = 1$$

cannot in general hold.

Furthermore the operator in the integrand depends on the total energy W. The procedure of Lévy and Klein [29] to eliminate the energy dependence brings about an interaction which is in general non-Hermitian. This can be seen for instance in the lowest approximation; in this case the integral equation for the $a_0^0(\lambda)$ is

$$(2.11) \qquad [W - E_0^0(\lambda)] a_0^0(\lambda) \quad \sum_{\mu\mu'} \frac{0, 0, \lambda \, | H' \, | \, 1, 0, \mu \quad 1, 0, \mu \, | H'' \, | 0, 0, \mu'}{W - E_1^0(\mu)} \quad a_0^0(\mu') \,.$$

The energy denominator is expanded in the following way:

$$(2.12) \qquad [W - E_1^0(\mu)]^{-1} = - \frac{1}{E_1^0(\mu) - E_0^0(\mu')} \sum_{n=0}^{\infty} \left(\frac{W - E_0^0(\mu')}{E_1^0(\mu) - E_0^0(\mu')} \right)^n \,.$$

Substituting this expansion into (2.11) we can use a method of successive interaction which consists in substituting $(W - E_0^0(\mu'))a_0^0(\mu')$ in the second member of the equation with its value given by the equation itself in the preceding iteration. One obtains then an integral equation for the a_0^0 with a kernel no longer depending on the energy and having the form of a power series expansion in the coupling constant. The second and fourth order interactions are, written in a symbolic form,

$$(2.13) \qquad \begin{cases} V_2 = -H_{01} H_{10}^\times \,, \\ V_4 = H_{01}(H_{10}^\times H_{01} H_{10}^\times)^\times \,, \end{cases}$$

where H_{10}, H_{01} couple the zero-meson and one-meson subspaces by creating and annihilating one meson respectively; clearly it is

$$(2.14) \qquad H_{01}^\dagger = H_{10};$$

$H_{01}^\times$ and $H_{10}^\times$ are defined by

$$(2.15) \qquad \begin{cases} \langle \mu' | H_{01}^\times | \mu \rangle = \langle \mu' | H_{01} | \mu \rangle (E_0^0(\mu') - E_1^0(\mu))^{-1} \,, \\ \langle \mu | H_{10}^\times | \mu' \rangle = \langle \mu | H_{10} | \mu' \rangle (E_1^0(\mu) - E_0^0(\mu'))^{-1} \,, \end{cases}$$

[29] A. KLEIN: Phys. Rev., **89**, 1158 (1953).

(O , with O any operator, has an exactly similar meaning). V_2 and V of (13) are not Hermitian, since

$$(2.16) \qquad (H_{01}^{\times})^{\dagger} = - H_{10}^{\rangle}, \qquad H_{01}^{\dagger} = H_{10} .$$

They turn out to be Hermitian when retardation effects are neglected, at least in cases of particular interest. Nevertheless the objection of their non-hermiticity in principle remains.

FELDMAN [30], making use of the formalism of canonical transformations, showed that it is possible to decouple the amplitude a_0^0 from all other amplitudes up to any desired order of the coupling constant by means of a succession of canonical transformations. This procedure will have physical meaning when the conditions, stated in the introduction, which make valid the description of the interaction by an ordinary potential, are verified. One obtains then an equation for a_0^0 with a Hermitian kernel which may be interpreted as a Schrödinger equation for the two nucleons.

The results of Feldman's method with the ps coupling Hamiltonian in the fixed source model are, with terms up to the order $(\mu/2M)^3$, the following:

$$(2.17) \qquad V_2 = \frac{g_1^2}{4\pi}\left(\frac{1}{2K}\right)^2 (\underset{\rightarrow}{\tau}^{(1)} \cdot \underset{\rightarrow}{\tau}^{(2)})(\boldsymbol{\sigma}^{(1)} \cdot \nabla)(\boldsymbol{\sigma}^{(2)} \cdot \nabla)\frac{\exp[-xr]}{r} =$$

$$= \mu c^2 \frac{g_1^2}{4\pi\hbar c}\left(\frac{\mu}{2M}\right)^2 (\underset{\rightarrow}{\tau}^{(1)} \cdot \underset{\rightarrow}{\tau}^{(.)})\left[\frac{1}{3}(\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(.)}) + \frac{1}{3}S_{12}\left(1 + \frac{3}{x} + \frac{3}{x^2}\right)\right]\frac{\exp[-x]}{x} , \text{(*)}$$

[30] D. FELDMANN: *Phys. Rev.*, **98**, 1456 (1953).

(*) In addition to the terms listed there is a term

$$-\mu c^2 \frac{g_1^2}{4\pi\hbar c}\left(\frac{\mu}{2M}\right)^3 (\underset{\rightarrow}{\tau}^{(1)} \cdot \underset{\rightarrow}{\tau}^{(2)})(\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)})\frac{4\pi}{3x^3}\delta(\boldsymbol{r}) ;$$

obviously it will be omitted since the validity of a theoretical potential is limited to the region $x > x_{\min}$.

From an historical point of view it must be pointed out that the singularity $1/x^3$ of the potential V_2 was viewed as a serious difficulty and various devices were contrived in order to overcome it. Among these we quote the mixed theory of Rosenfeld and Møller [31], Bethe's cut-off [32] and the non-linear theories [33]. In the light of the arguments stated at the beginning it appears clear that this difficulty fades when one bears in mind that a theoretical potential cannot hold for nucleon separations $x < x_{\min}$.

[31] C. MØLLER and L. ROSENFELD: *Kgl. Dan. Vid. Selskab.*, **17**, 8 (1940); C. MØLLER: *Kgl. Dan. Vid. Selskab*, **18**, 6 (1941). See also J. M. JAUCH and N. HU: *Phys. Rev.*, **65**, 289 (1944).

[32] H. A. BETHE: *Phys. Rev.*, **55**, 1261 (1939).

[33] L. I. SCHIFF: *Phys. Rev.*, **84**, 1, 10 (1951); **86**, 856 (1952).

(2.18) $$V_4 = V_{4a} + V_{4b},$$

(2.18a) $$V_{4a} = -3\mu c^2 \lambda^2 \left(\frac{g_1^2}{4\pi\hbar c}\right)^2 \left(\frac{\mu}{2M}\right)^2 \left[\frac{2}{\pi}\frac{K_1(2x)}{x^2}\right],$$

(2.18b) $$V_{4b} = 6\mu c^2 \lambda \left(\frac{g_1^2}{4\pi\hbar c}\right)^2 \left(\frac{\mu}{2M}\right)^3 \left(1 + \frac{1}{x}\right)^2 \frac{\exp[-2x]}{x^2},$$

where $x = \varkappa r$; $K_n(x)$ is a modified Hankel function of the first kind: $K_n(x) = (\pi/2)i^{n+1}H_n^{(1)}(ix)$. The potential V_2 and V_4 of (17) and (18) are in agreement with those previously obtained by LÉVY and KLEIN.

The term V_{4a} has its origin in the two-pair processes; the term V_{4b} in the one-pair processes; the no-pair processes bring about a term of the order $(\mu/2M)^4$ which is therefore here neglected.

The potential $V = V_2 + V_4$ given by (17) and (18) is wholly inadequate, on account of the near-cancellation between the attractive term V_{4a} and the repulsive term V_{4b}.

The pv coupling Hamiltonian yields in the fixed source model the T.M.O. potential (see (2.43), (2.44)).

An equivalent method to overcome the difficulties above has been used by FUKUDA, SAWADA and TAKETANI ([34]) (*).

Denoting by P the projection operator on the two-nucleon subspace, we may write

(2.19) $$|\Phi\rangle = P|\Phi\rangle + (1 - P)|\Phi\rangle.$$

Projecting the eq. (1.1') on the two subspaces one obtains the following two equations:

(2.20) $$(W - H_0)P|\Phi\rangle = PH'P|\Phi\rangle + PH'(1 - P)|\Phi\rangle,$$

(2.21) $$(W - H_0)(1 - P)|\Phi\rangle = (1 - P)H'(1 - P)|\Phi\rangle + (1 - P)H'P|\Phi\rangle.$$

A formal solution of (1.1') may be written

(2.22) $$|\Phi\rangle = J(W)P|\Phi\rangle,$$

with

(2.23) $$J(W) = \frac{1}{1 - (1/(W - H_0))(1 - P)H'}.$$

([34]) N. FUKUDA, K. SAWADA and M. TAKETANI: *Progr. Theor. Phys.*, **12**, 156 (1954).

(*) A very similar procedure was devised by OKUBO ([35]).

([35]) S. OKUBO: *Progr. Theor. Phys.*, **12**, 603 (1954).

The energy dependence of $J(W)$ may be eliminated making use of the equations of motion.

It is evident that

$$(2.24) \qquad \langle \Phi P J^\dagger | J P \Phi \rangle = \langle \Phi | \Phi \rangle = 1 \, .$$

This suggests a linear transformation

$$(2.25) \qquad |\chi\rangle = (J^\dagger J)^{\frac{1}{2}} P | \Phi \rangle$$

in the two-nucleon subspace, assumed to be complete; the $|\chi\rangle$ are correctly normalized

$$\langle \chi_a | \chi_b \rangle = \delta_{ab}$$

and satisfy the equation

$$(2.26) \quad (W - H_N) | \chi \rangle = [P(J^\dagger J)^{-\frac{1}{2}} P J^\dagger (H_0 + H') J P (J^\dagger J)^{-\frac{1}{2}} P - P H_0 P] \rangle | \chi \rangle \, ,$$

which may be interpreted as a correct Schrödinger equation for the two physical nucleons. At the static limit ($H_N = 0$) the interaction kernel yields the static potential

$$(2.27) \qquad V = P \frac{1}{J^\dagger J} P P J^\dagger (H_M + H') J P \, .$$

The factor $P(1/J^\dagger J)P$ is called the « probability operator » inasmuch as its expectation value represents the probability for two « physical » nucleons to be bare

$$(2.28) \qquad \langle \chi | P \frac{1}{J^\dagger J} P | \chi \rangle = \langle \chi (J^\dagger J)^{-\frac{1}{2}} | (J^\dagger J)^{-\frac{1}{2}} \chi \rangle = \langle \Phi P | P \Phi \rangle \, .$$

It is subject to the physical condition

$$(2.29) \qquad 0 < \langle \chi | \frac{1}{J^\dagger J} | \chi \rangle < 1 \, .$$

The factor

$$PJ^\dagger (H_M + H') J P$$

is called the « normal part ».

In order to obtain the potential up to the fourth order in the coupling constant one must expand (2.27) into a power series in the coupling constant.

Different forms of the potential obtain depending on the way of treating the two factors. Putting $1/(J^\dagger J) - 1$ and expanding the numerator up to the fourth order one obtains the Brueckner and Watson potential (see (2.74), (2.75))

Expanding the factor $1/J^\dagger J$ up to the second order and the normal part up to the fourth order (in the expansion of $1/J^\dagger J$ the first term is 1, the second term is of second order in the coupling constant) one obtains the T.M.O potential (see (2.43), (2.44)). In this case however the condition (2.29) becomes

(2.30) $$0 < \langle \chi | 1 - PH' \frac{1 - P}{(-H_0)^2} H' P | \chi \rangle < 1 .$$

Now while (2.29) is automatically satisfied, (2.30) like any condition where $(J^\dagger J)^{-1}$ is approximated by a series expansion cut at a certain order, must be explicitly imposed as a supplementary condition and can thus be a useful guide in the discussion of the region of validity of the potential obtained.

F.S.T. propose as static nuclear potential

(2.31) $$V = \frac{\left\langle H' \dfrac{1 - P}{- H_0} H' \right\rangle + \left\langle H' \dfrac{1 - P}{- H_0} H' \dfrac{1 - P}{- H_0} H' \dfrac{1 - P}{- H_0} H' \right\rangle}{\left\langle 1 + H' \dfrac{1 - P}{(- H_0)^2} H' \right\rangle}$$

($\langle ... \rangle$ means meson vacuum expectation value), *i.e.* with the denominator expanded up to second order and the normal part up to fourth order. The argument for this prescription is the following; for the fixed source model of the neutral scalar theory, the exact solution is known [36]; this solution is reproduced by the method above only if the prescription is adopted. Furthermore the presence of $J^\dagger J$ in the denominator reduces the singularity of the normal part, as can be intuitively seen considering that a strong singularity arises when the meson cloud around the nucleon is intense and, accordingly, the probability for the nucleons to be bare becomes much smaller.

It must be pointed out that the explicit evaluation of the potential taking into account the probability factor, is very sensitive to the renormalization procedure (cut-off or other); the validity of condition (2.29) (or an analogous one in which an approximate form of $(J^\dagger J)^{-1}$ is used) requires the inclusion of the self-energy terms which cannot be simply ignored.

The detailed calculation along the lines set up by F.S.T. has been carried out by INOUE, MACHIDA, TAKETANI and TOYODA [37], starting from the pv

[36] See *e.g.* WENTZEL: *Quantum theory of fields* (New York, 1949), p. 47.
[37] K. INOUE, S. MACHIDA, M. TAKETANI and S. TOYODA: *Prog. Theor. Phys.*, **15**, 122 (1956).

Hamiltonian (1.22) and choosing

$$c(k) = \exp\left[-\frac{k^2}{k_{\max}^2}\right],$$

for the Fourier transform of the source function describing the nucleon.

The resulting potential is

$$(2.32) \quad V = \frac{1}{3}\frac{g_2^2}{4\pi}\frac{1}{\varkappa^2}\frac{2}{\pi}(\underset{\rightarrow}{\tau^{(1)}}\cdot\underset{\rightarrow}{\tau^{()}})\int_0^\infty c^2(k_1)[(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(\cdot)})(2A_1+B_1)-S_{12}(A_1-B_1)]\,\frac{k_1}{\omega_1^2}\,\mathrm{d}k_1 -$$

$$-\left(\frac{g_2^2}{4\pi}\right)^2\frac{1}{\hbar c\varkappa^4}\left(\frac{2}{\pi}\right)^2\int_0^\infty\int_0^\infty\left(\frac{2}{\omega_1}+\frac{1}{\omega_2}\right)[2(\underset{\rightarrow}{\tau^{(1)}}\cdot\underset{\rightarrow}{\tau^{(2)}})(B_1B_2+2A_1A_2) +$$

$$+2(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(\cdot)})(A_1B_2+B_1A_2+A_1A_2)-S_{12}(A_1B_2+B_1A_2-2A_1A_2)]\cdot$$

$$\cdot\frac{c^2(k_1)c^2(k_2)k_1k_2}{\omega_1^2\omega_2(\omega_1+\omega_2)}\,\mathrm{d}k_1\,\mathrm{d}k_2\cdot\left\{1+3\Delta_2'-\frac{1}{3}\frac{g_2^2}{4\pi\hbar c}\frac{1}{\varkappa^2}\frac{2}{\pi}(\underset{\rightarrow}{\tau^{(1)}}\cdot\underset{\rightarrow}{\tau^{()}})\cdot\right.$$

$$\left.\cdot\int_0^\infty c^2(k_1)[(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(\cdot)})(2A_1+B_1)-S_{12}(A_1-B_1)]\,\frac{k^1}{\omega_1^3}\,\mathrm{d}k_1\right\}^{-1},$$

where

$$\begin{cases} A_i = \frac{1}{r^3}\,(k_i r\cos k_i r - \sin k_i r)\,, \\[2mm] B_i = \frac{1}{r^3}\,(2\sin k_i r - 2k_i r\cos k_i r - k_i^2 r^2 \sin k_i r)\,, \qquad (i=1,2)\,, \\[2mm] \omega = \sqrt{\left(\frac{\mu c}{\hbar}\right)^2+\boldsymbol{k}^2}\,, \end{cases}$$

and Δ_2' is a renormalization constant (*).

(*) Following the method of Chew [38] I.M.T.T. carry out the renormalization by the formula

$$g_2 = G_2 Z_1 Z_2^{-1}\,,$$

where G_2 is the renormalized coupling constant and Z_1, Z_2 are renormalization constants for the vertex and the nucleon propagator, respectively; up to the second order in the coupling constant one obtains

$$Z_1 = 1-\frac{1}{6}\Delta_2'\,, \qquad Z_2 = 1-\frac{3}{2}\Delta_2'\,, \quad\text{with}\quad \Delta_2' = \left(\frac{G_2}{\varkappa}\right)^2\frac{1}{\hbar c(2\pi)^3}\int\frac{c^2(k)k^2}{\omega^3}\,\mathrm{d}\boldsymbol{k}\,.$$

[38] G. F. CHEW: *Phys. Rev.*, **95**, 1669 (1954).

The most serious objection which can be moved to the T.D. method is that within its framework the renormalization program cannot be carried out in an unambiguous way. Even Cini's [39] covariant renormalization method is not free of ambiguities [40]; nor completely renormalizable is the « new Tamm-Dancoff method » proposed by DYSON [41]. The essence of this method is the re-establishment of symmetry between emission and absorption (or between particles and antiparticles), which is lacking in the old T.D. method, through the definition of a new probability amplitude

$$ a(N', N) = \langle \Phi_0 \, C(N') A(N) \Phi \rangle \,, $$

where $|\Phi\rangle$ is the actual state of the system, $|\Phi_0\rangle$ the true vacuum, $A(N)$ is the product of free particle annihilation operators which annihilates the particles specified by N, and $C(N')$ is the product of the corresponding creation operators.

The new T.D. method has been applied to the nuclear force problem by TAYLOR [42] and KURŞUNOĞLU [43] and at the static limit yields the same results of the original T.D. method.

2˙2. *S-matrix methods.* – The S-matrix formalism is one of the most widely used for the construction of the potential. In the past this formalism was favoured because it supplied the tool for the treatment of renormalization problems and connected subjects and more recently because it has opened the gate to the introduction of dispersion techniques in this field.

This shift of emphasis is reflected in the use of the formalism by the various authors in the course of time, as it will be seen from the following account, in which a historic line will be followed.

We point out from the outset that the potential yielded by these methods is a potential for scattering states.

The common feature of the definition of the potential in the framework of the S-matrix is the identification of certain quantities describing the scattering of two particles in particle wave-mechanics, such as the s-matrix elements, the t-matrix elements, the k-matrix elements, with the analogous quantities such as the S-matrix elements, the T-matrix elements, the K-matrix elements, describing the same situation in quantum field theory.

The actual construction of the potential is carried out in different ways

[39] M. CINI: *Nuovo Cimento*, **10**, 526, 614 (1953).
[40] F. J. DYSON: *Phys. Rev.*, **91**, 421 (1953).
[41] F. J. DYSON: *Phys. Rev.*, **90**, 994 (1953); **91**, 1543 (1953).
[42] J. C. TAYLOR: *Phys. Rev.*, **95**, 1313 (1954); **96**, 438 (1954).
[43] B. KURŞUNOĞLU: *Phys. Rev.*, **96**, 1690 (1954).

by the various authors, depending on which quantities are chosen for the identification and how the latter is performed.

We briefly list the quantities of interest and the relations establishing a link between them, which are written in the center-of-mass system.

From an S-matrix element between nucleon states $|a\rangle$ and $|b\rangle$ a T-matrix element is defined by:

$$(2.33) \qquad \langle a\,|\,S\,|\,b\rangle = \delta_{ab} - 2\pi i\,\delta(E_a - E_b)\langle a\,|\,T\,|\,b\rangle\,.$$

In virtue of the unitarity of the S-matrix, T satisfies the following equation:

$$(2.34) \qquad 2\pi \sum_c \delta(E_a - E_c)\langle a\,|\,T^\dagger\,|\,c\rangle\langle c\,|\,T\,|\,b\rangle = i\langle a\,|\,T - T^\dagger\,|\,b\rangle\,.$$

Scattering processes may also be described by the hermitean operator $\mathscr{K}$ which is linked to the S-matrix by the equation

$$(2.35) \qquad S = \frac{1 - \frac{1}{2}i\mathscr{K}}{1 + \frac{1}{2}i\mathscr{K}}\,.$$

Besides the $\mathscr{K}$-matrix, the reaction matrix K is also used:

$$(2.36) \qquad \langle a\,|\,K\,|\,b\rangle = 2\pi\,\delta(E_a - E_b)\langle a\,|\,\mathscr{K}\,|\,b\rangle\,.$$

Completely analogous quantities appear in particle wave mechanics and will be indicated by small letters.

We finally note that t satisfies the Lippmann-Schwinger equation [44]:

$$(2.37) \qquad \langle a\,|\,t\,|\,b\rangle = \langle a\,|\,v\,|\,b\rangle + \sum_c \frac{\langle a\,|\,v\,|\,c\rangle\langle c\,|\,t\,|\,b\rangle}{E_b - E_c + i\eta}\,.$$

The old-fashioned construction of the potential was carried out according to the following procedure; the Lippmann-Schwinger equation was solved by a perturbative method:

$$(2.38) \qquad t = t_2 + t_4 + \ldots\,,$$

where

$$(2.39) \qquad \langle a\,|\,t_2\,|\,b\rangle = \langle a\,|\,V_2\,|\,b\rangle\,,$$

$$(2.40) \qquad \langle a\,|\,t_4\,|\,b\rangle = \langle a\,|\,V_4\,|\,b\rangle + \sum_c \frac{\langle a\,|\,V_2\,|\,c\rangle\langle c\,|\,V_2\,|\,b\rangle}{E_b - E_c + i\eta}\,,$$

[44] See *e.g.* L. HULTHÉN and M. SUGAWARA: *Handbuch der Physik*, vol. XXXIX Berlin, 1957), p. 135. For the formal theory of scattering see also: A. S. DAVȲDOV: *Teoriya atomnogo yadra* (Moscow, 1958), Chapt. IX.

etc.; each of these terms was then equated to the static limit $(\mu/M \to 0)$ of the field theoretical terms T_2, T_4, ..., arising from the one-pion, two-pion, etc., exchange graphs:

$$(2.41) \qquad \langle a\,|\,t_2\,|\,b\rangle = \langle a\,|\,T_2\,|\,b\rangle_{\text{stat}}\,,$$

$$(2.42) \qquad \langle a\,|\,t_4\,|\,b\rangle = \langle a\,|\,T_4\,|\,b\rangle_{\text{stat}}\,,$$

etc..

A few remarks are in order at this point. We note first that the T-matrix is defined only on the energy shell. While the matrix elements on the energy shell are sufficient for the definition of the one-pion exchange potential, matrix elements off the energy shell are necessary for more than one-pion exchange, as can be seen for the two-pion exchange from eq. (2.40). The method can therefore yield a whole class of potentials arising from the same S-matrix.

Moreover the crude identification assumed by the method ignored certain difficulties: in particular the static limit was carried out on the T-matrix elements and these elements were then equated to the t-matrix elements without examining if the necessary conditions were satisfied. Now, as we have noted in the introduction, the static limit is not justified; moreover the t-matrix elements satisfy certain conditions, which are not satisfied in general by the T-matrix elements.

In spite of its shortcomings the method has been widely used in the past by various authors, some of which have also undertaken a calculation of radiative corrections of various orders and an assessment of their effect on the one-pion and two-pion exchange potentials.

As examples of these researches we quote the contribution of WATSON and LEPORE [45], who by means of an evaluation of the lowest order radiative corrections to the one-pion potential, showed that these reduce to coupling constant and mass renormalization and that of HIIDA, IWADARE and MACHIDA [46], who reached the same conclusion but taking into account radiative corrections of all orders; the calculation of the potential by I. SATO [47] for both the ps-ps coupling (giving the Lévy-Klein potential) and the ps-pv coupling (giving the T.M.O. potential) and the compact formulation of the problem by IMAMURA [48]. The best known result of this group of research was the T.M.O. potential [49] which enjoyed a certain fame for several years and which results from the explicit calculation of the expression of NAMBU [50],

[45] K. M. WATSON and J. V. LEPORE: Phys. Rev., 76, 1157 (1949).
[46] K. HIIDA, J. IWADARE and S. MACHIDA: Progr. Theor. Phys., 15, 189 (1956).
[47] I. SATO: Prog. Theor. Phys., 10, 323 (1953).
[48] T. IMAMURA: Progr. Theor. Phys., 13, 183 (1955).
[49] M. TAKETANI, S. MACHIDA and S. ONUMA: Progr. Theor. Phys., 7, 45 (1952).
[50] Y. NAMBU: Progr. Theor. Phys., 5, 614 (1950).

using the ps-pv coupling:

$$(2.43) \qquad V_2 = \mu c^2 \, \frac{g_2^2}{4\pi\hbar c} \, (\underset{\rightarrow}{\tau}^{(1)} \cdot \underset{\rightarrow}{\tau}^{(2)}) \cdot$$

$$\cdot \left[\frac{1}{3} \, (\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)}) \, \frac{\exp[-x]}{x} + \frac{1}{3} \, S_{12} \left(1 + \frac{3}{x} + \frac{3}{x^2} \right) \frac{\exp[-x]}{x} \right].$$

$$(2.44) \qquad V_4 = \mu c^2 \left(\frac{g_2^2}{4\pi\hbar c} \right)^2 [(\underset{\rightarrow}{\tau}^{(1)} \cdot \underset{\rightarrow}{\tau}^{(2)}) \, U_\tau(x) + (\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)}) \, U_\sigma(x) + S_{12} U_T(x)],$$

$$(2.44a) \qquad U_\tau(x) = -\frac{8}{\pi} \left\{ \left(\frac{1}{x} + \frac{23}{4x^3} \right) K_0(2x) + \left(\frac{3}{x^2} + \frac{23}{4x^4} \right) K_1(2x) \right\},$$

$$(2.44b) \qquad U_\sigma(x) = \frac{8}{\pi} \left\{ \frac{3}{x^3} \, K_0(2x) + \left(\frac{2}{x^2} + \frac{3}{x^4} \right) K_1(2x) \right\},$$

$$(2.44c) \qquad U_T(x) = -\frac{8}{\pi} \left\{ \frac{3}{x^3} \, K_0(2x) + \left(\frac{1}{x^2} + \frac{15}{4x^4} \right) K_1(2x) \right\}.$$

A more elaborate version was given by HENLEY and RUDERMAN [51], with an attempt of a calculation of multiple scattering corrections; they showed that it is possible to write the S-matrix element for the two-nucleon scattering in terms of matrix elements for meson-nucleon scattering.

Using the same idea, MIYAZAWA [52], treating nucleons as fixed sources coupled to the meson field by pv coupling, was able to express the potential in terms of the experimental quantities for meson-nucleon scattering, making use of dispersion relations to define the meson-nucleon scattering amplitude for all values of the energy from minus infinity to plus infinity (real scattering obviously occurs only for energies larger than the pion rest mass).

The potential obtained by MIYAZAWA, which was explicitly calculated by KONUMA, MIYAZAWA and OTSUKI [53], is remarkably different from those previously obtained; we do not quote it here on account of its length.

More recently the problem has been formulated on a more critical basis. One point which has been definitely clarified is that the static limit is completely unjustified. In spite of this, it has been shown that the construction of a potential up to two-pion exchange is nevertheless still possible. The discussion of the possibility of the definition of the potential and of its limits and the actual construction of this can be done with various techniques; in the

[51] E. M. HENLY and M. A. RUDERMAN: *Phys. Rev.*, **92**, 1036 (1953).
[52] H. MIYAZAWA: *Phys. Rev.*, **104**, 1741 (1956).
[53] M. KONUMA, H. MIYAZAWA and S. OTSUKI: *Progr. Theor. Phys.*, **19**, 17 (1958).

introduction we have given an account of one of these more modern approaches due to BRAUN ([11]). The construction of the potential is based, following the suggestion of NOVOZHILOV, instead of on the perturbative procedure on the Lippmann-Schwinger equation, on the use of a non-linear equation which results from a formal solution of that equation, which gives the potential explicitly in terms of the t-matrix. As we have seen, it is actually possible, although not in a unique way, to obtain from quantum field theory certain $\widetilde{T}$-matrices which are identical, under the creation threshold and on the energy shell, with the T-matrix and which possess the necessary requirements in order to be inserted in place of the t-matrix in the non-linear equation, thus yielding an expression for the potential. We record here the formal expression of the « potential » thereby obtained:

$$(2.45) \qquad V_{\alpha\beta} = \{[1+N]^{-\frac{1}{2}} \widetilde{V}(E_\alpha, E_\beta)[1+N]^{-\frac{1}{2}}\}_{\alpha\beta},$$

$$(2.45a) \qquad \widetilde{V}_{\gamma_1\gamma_2}(E_\alpha, E_\beta) = \langle \Phi_{\gamma_1}|H - \frac{E_\alpha + E_\beta}{2}|\Phi_{\gamma_2}\rangle +$$
$$+ \sum_{\Gamma \neq \gamma} \frac{T_{\gamma_1\Gamma}T^\dagger_{\Gamma\gamma_2}(E - \frac{1}{2}(E_\alpha + E_\beta))}{(E_{\gamma_1} - E_\Gamma - i\eta)(E_\Gamma - E_{\gamma_2} - i\eta)}.$$

We recall that this is a non-local potential; we refer to the introduction for a discussion of this feature.

An alternative formulation of the problem is due to GUPTA ([54]), who used the $\mathscr{K}$ operator linked to the S-matrix by eq. (2.35). If $\mathscr{K}$ is expanded in the following way

$$(2.46) \qquad \mathscr{K} = \sum_{n=1}^{\infty} \mathscr{K}_n$$

and this expansion is introduced step by step in (2.35), one obtains a « unitary expansion » of the S-matrix, since it is unitary at any stage, unlike the usual Dyson expansion

$$(2.47) \qquad S = 1 + \sum_{n=1}^{\infty} S_n,$$

which is unitary only as a whole.

The $\mathscr{K}_n$ can be easily expressed in terms of the S_n; we record here only the expression for $\mathscr{K}_2$ and $\mathscr{K}_4$ for the two-nucleon scattering:

$$\mathscr{K}_2 = iS_2,$$

$$\mathscr{K}_4 = \tfrac{1}{2} i(S_4 - S_4^\dagger).$$

([54]) S. N. GUPTA: Phys. Rev., 117, 1146 (1960); Nuovo Cimento, 18. 823 (1960).

Taken the unitary expansion of S in first approximation an effective inter-action Hamiltonian H_{eff} can be defined such that the contribution of H_{eff} to the unitary expansion in first approximation is equal to the total scattering operator

$$\frac{1 + \frac{1}{2}\left(- i/\hbar\right)\int H_{\text{eff}}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t}{1 - \frac{1}{2}\left(- i/\hbar\right)\int H_{\text{eff}}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t} \quad \frac{1 - \frac{1}{2}i\mathscr{K}}{1 + \frac{1}{2}i\mathscr{K}} \, ,$$

which immediately yields

$$(2.48) \qquad\qquad \mathscr{K} = (1/\hbar)\int H_{\text{eff}}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \, .$$

The « potential » may be defined by expressing the effective interaction energy of the two nucleons, in the center-of-mass system, in the following way:

$$(2.49) \qquad \langle a\,|\int H_{\text{eff}}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\,|\,b\rangle = 2\pi\hbar\delta(E_a - E_b)\langle a\,|\,\mathscr{V}\,|\,b\rangle \, .$$

It is easily seen that its definition in terms of the $\mathscr{K}$-matrix is

$$(2.50) \qquad \langle a\,|\,\mathscr{K}\,|\,b\rangle = 2\pi\,\delta(E_a - E_b)\langle a\,|\,\mathscr{V}\,|\,b\rangle \, .$$

The one-pion and two-pion exchange potentials can be easily obtained from this taking the adiabatic limit of $\mathscr{K}_2$, $\mathscr{K}_4$.

The result of the calculation with ps-ps coupling for the one-pion exchange is in agreement with the usual expression; the two-pion exchange term is given by

$$(2.51) \qquad V_4 = - \left(\frac{g_1^2}{4\pi c\hbar}\right)^2 \mu c^2 [P(x) - (\vec{\tau}^{(1)}\cdot\vec{\tau}^{(2)})Q(x)] \, ,$$

where

$$(2.51a) \qquad P(x) = \frac{3M^2}{8\pi\mu^2}\int_0^1 \mathrm{d}u\int_0^u \mathrm{d}v\,\frac{v^3}{(u - v)^2(1 - u)^2}\,\frac{1}{f}\exp\left[- fx\right] ,$$

$$(2.51b) \qquad Q(x) = \frac{1}{2\pi}\int_0^1 \mathrm{d}u\int_0^u \mathrm{d}v\,\frac{v}{(u - v)(1 - u)}\,\frac{1}{x}\exp\left[- fx\right] - \frac{2}{3}\,P(x) ,$$

with

$$(2.51c) \qquad\qquad f = \left[\frac{(M^2/\mu^2)v^2 + (1 - v)}{(u - v)(1 - u)}\right]^{\frac{1}{2}} .$$

One might think of following a procedure analogous to that described above for the T-matrix consisting in identifying step by step the terms of the k-matrix of particle wave mechanics with the terms of the K-matrix of field theory. This would obviously result in a potential identical with that defined by means of the T-matrix but different, in general, from that obtained from Gupta's procedure. While the one-pion exchange term of Gupta is still identical with that of the previous method, the two-pion exchange term differs from that of the previous method by a principal part. It is easy to see that the two-pion exchange term in the potential defined by identifying K_4 with k_4 is

$$\langle a \,|\, V_4 \,|\, b \rangle = \langle a \,|\, K_4 \,|\, b \rangle - \sum_c \langle a \,|\, K_2 \,|\, c \rangle \, P \left(\frac{1}{E_b - E_c} \right) \langle c \,|\, K_2 \,|\, b \rangle \,,$$

while Gupta's two-pion exchange term is

$$\langle a \,|\, V_4^G \,|\, b \rangle = \langle a \,|\, K_4 \,|\, b \rangle \,.$$

On account of this fact Gupta's two-pion exchange potential contains, unlike the potential defined in the preceding way, only T-matrix elements on the energy shell, as can be seen by comparing the expression of the two potentials in terms of T-matrix elements:

$$\langle a \,|\, V_4 \,|\, b \rangle = \langle a \,|\, T_4 \,|\, b \rangle - \sum_c \frac{\langle a \,|\, T_2 \,|\, c \rangle \langle c \,|\, T_2 \,|\, b \rangle}{E_b - E_c + i\eta} \,,$$

$$\langle a \,|\, V_4^G \,|\, b \rangle = \langle a \,|\, T_4 \,|\, b \rangle + i\pi \sum_c \langle a \,|\, T_2 \,|\, c \rangle \, \delta(E_b - E_c) \langle c \,|\, T_2 \,|\, b \rangle \,.$$

In the foregoing expressions the adiabatic limit is to be intended whenever T- or K-matrix elements appear. In the old-fashioned methods use was made as a rule of the static limit; it can be shown that this is needless because the potential can be constructed up to two-pion exchange making use of the adiabatic limit only. That this is unjustified has been repeatedly emphasized and is shown in detail in Gupta's paper.

The recent interest for dispersion relations and techniques in theoretical physics has suggested a promising and powerful approach to the problem of the nuclear potential.

Although persisting difficulties still prevent the integral extension of the method to the case of nucleons with spin interacting through pseudoscalar symmetric meson field, the cases of the scalar nucleons interacting through neutral scalar meson field or charged scalar meson field can be exhaustively discussed [14].

We give here a brief account for the case of the neutral scalar field. Let

us assume that a dispersion relation at fixed momentum transfer

$$(2.52) \qquad t(E, \tau) = V(\tau) + \frac{1}{\pi} \int_0^\infty \frac{\operatorname{Im} t(E', \tau)}{E' - E - i\eta} \, \mathrm{d}E' \,,$$

(where τ is the momentum transfer) is available. From this the following relations are immediately obtained:

$$(2.53) \qquad t(E, \tau) = t(0, \tau) + \frac{1}{\pi} \int_0^\infty \frac{E}{E'} \frac{\operatorname{Im} t(E', \tau)}{E' - E - i\eta} \, \mathrm{d}E' \,,$$

$$(2.54) \qquad V(\tau) = t(0, \tau) - \frac{1}{\pi} \int_0^\infty \frac{\operatorname{Im} t(E', \tau)}{E'} \, \mathrm{d}E' \,.$$

Beside these we consider the condition of unitarity of the s-matrix:

$$(2.55) \qquad i\langle \boldsymbol{p}' | t^\dagger - t | \boldsymbol{p} \rangle + \frac{2\pi}{\hbar^3} \int \langle \boldsymbol{p}' | t^\dagger | \boldsymbol{p}'' \rangle \, \delta\left(\frac{\boldsymbol{p}''^2}{M} - \frac{\boldsymbol{p}'^2}{M} \right) \langle \boldsymbol{p}'' | t | \boldsymbol{p} \rangle \, \mathrm{d}\boldsymbol{p} = 0 \,.$$

On the other hand for the field transition amplitude G, which differs from the previously defined T by a factor $M^2 c^4 / E^2$, which is about 1 for $\boldsymbol{p}^2 \ll M^2 c^2$, a dispersion relation holds, and a unitarity relation, which for energies lower enough than the threshold, may be written in the following way

$$(2.56) \qquad G(E, \tau) = G(0, \tau) + \frac{1}{\pi} \int_0^\infty \frac{E}{E'} \frac{\operatorname{Im}{}^{\mathrm{el}} G(E, \tau)}{E' - E - i\eta} \, \mathrm{d}E' + O\left(\frac{E}{E_{\mathrm{thresh}}} \right) ,$$

$$(2.57) \qquad i\langle \boldsymbol{p}' | G^\dagger - G | \boldsymbol{p} \rangle_{\mathrm{el}} + \frac{2\pi}{\hbar^3} \int \langle \boldsymbol{p}' | G^\dagger | \boldsymbol{p}'' \rangle \, \delta\left(\frac{\boldsymbol{p}'^2}{M} - \frac{\boldsymbol{p}''^2}{M} \right) \langle \boldsymbol{p}'' | G | \boldsymbol{p} \rangle \, \mathrm{d}\boldsymbol{p}'' = 0 \,,$$

where $\operatorname{Im}{}^{\mathrm{el}} G$ is the contribution to the imaginary part of G coming from elastic processes.

If one identifies $t(0, \tau)$ with $G(0, \tau)$, it is seen that $t(E, \tau)$ on the one hand and $G(E, \tau)$ on the other satisfy (when neglecting $O(E/E_{\mathrm{thresh}})$) the same dispersion and unitarity relations. This identification provides therefore a reasonable basis for the definition of a potential from quantum field theory. Substituting $G(0, \tau)$ for $t(0, \tau)$ and solving the non-linear set of eq. (2.53) and (2.55), eq. (2.54) provides a definition for the potential.

This program may be carried out for scalar nucleons interacting through a neutral scalar meson field. In fact for this case a dispersion relation of the

type (2.52), known as Khuri's relation, is available and a method for solving explicitly the set of eq. (2.53) and (2.55) may be devised; the result is a potential which is a superposition of Yukawa potentials with a spectral function giving the mass spectrum, when neglecting the so-called rescattering corrections (*). The one-pion exchange potential is the well-known Yukawa potential; the two-pion exchange potential has an expression such that it would vanish in the static limit, if this were justified.

The program may also be carried out when the scalar spinless nucleons interact through a charged scalar meson field.

In fact $t(E, \tau)$ may then be split into two parts, each of which satisfies a Khuri relation; an analogous splitting for the field theoretical $G(E, \tau)$ enables an identification to be performed in a way similar to that described above.

When the spinor character of nucleons is taken into account, the program outlined cannot be carried out because in this case the necessary dispersion relations are not yet available. One might be confident in the eventual availability of these relations and therefore one might postulate them; however the transition amplitude is made up of five terms corresponding to five spinor invariants, the choice of which is however not unique; while one might expect the dispersion relations to hold for a particular choice of these only and not for an arbitrary one. Under these circumstances CHARAP and TAUSNER ([56]), returning to the old definition based on the step identification of the t-matrix elements and the T-matrix elements, have shown that the condition that the two-pion exchange potential be independent of the energy in the adiabatic limit leads to restrictions on the choice of the spinor invariants. This condition may therefore be deemed a useful guide for the choice of these.

2˙3. *Method of Brueckner and Watson* ([57]). – The method employs the Lippman and Schwinger integral equation and the technique developed by CHEW, GOLDBERGER and WATSON to treat this equation ([58]).

Let $|\chi\rangle$ be an eigenstate of the free Hamiltonian. A scattering solution of eq. (1.1′) may be written

$$(2.58) \qquad\qquad |\Phi\rangle = \Omega\,|\chi\rangle \,,$$

with Ω the Möller wave matrix, satisfying the Lippamnn and Schwinger

(*) For recent progress in the treatment of these corrections see CINI and FUBINI ([55]).

([55]) M. CINI and S. FUBINI: *Ann. Phys.*, **10**, 352 (1960).

([56]) J. M. CHARAP and M. J. TAUSNER: *Nuovo Cimento*, **18**, 316 (1960)).

([57]) K. A. BRUECKNER and K. M. WATSON: *Phys. Rev.*, **90**, 699 (1953); **92**, 1023 (1953).

([58]) G. F. CHEW and M. L. GOLDBERGER: *Phys. Rev.*, **87**, 778 (1952); K. M. WATSON: *Phys. Rev.*, **89**, 575 (1953).

equation

(2.59) $$\Omega = 1 + \frac{1}{W - H_0 + i\eta} H' \Omega \, ,$$

whose formal solution is

(2.60) $$\Omega = 1 + \frac{1}{W - H_0 - H' + i\eta} H' \, .$$

In the case of nuclear forces we are interested in the diagonal part of Ω, which refers to the scattering of two-nucleons, without creation or annihilation of real mesons, and we need to express this part in terms of operators which are diagonal in occupation number.

This expression for Ω, which we denote by Ω_s, is reached by an inductive procedure which we relate for the case of an interaction Hamiltonian assumed to be diagonal in nucleon occupation number and non-linear in meson field variables Φ. This Hamiltonian can be written

(2.61) $$H' = H_1 + H_2 \, ,$$

where H_1 is odd in Φ and H_2 is even in Φ. We then obtain for Ω

(2.62) $$\Omega = 1 + \frac{1}{A - H_1 - H_2} (H_1 + H_2) =$$

$$= 1 + \frac{1}{A - H_2 - D_0} (H_2 + D_0) + \frac{1}{A - H_2 - D_0} H_1 \left[1 + \frac{1}{A - H_2} H_2 \right] , \quad (*)$$

where

(2.63) $$D_0 = H_1 \frac{1}{A - H_2} H_1 \, , \qquad A = W - H_0 \quad (**) \, .$$

The last term of (2.62), odd in H_1 (and thus in Φ), is non-diagonal in meson

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(*) From the operator relation

$$\frac{1}{A - (H_1 + H_2)} (H_1 + H_2) = \frac{1}{A - L} (H_1 + H_2) + \frac{1}{A - L} L \, ,$$

$$L = (H + H_2) \frac{1}{A} (H_1 + H_2) \, ,$$

using the identity

$$F^{-1} - G^{-1} = F^{-1}(G - F)G^{-1} = G^{-1}(G - F)F^{-1} \, ,$$

the last expression of (62) is obtained.

(**) $i\eta$ may be omitted when not treating processes involving real mesons.

occupation number and therefore does not contribute to the diagonal part of Ω, which we require. Accordingly it may be dropped.

We have then

$$(2.64) \qquad \Omega_s^{(0)} = \left[1 + \frac{1}{A - H_2 - D_0} (H_2 + D_0) \right]_{\text{diag}} .$$

We define

$$(2.65) \qquad \begin{cases} v_0 & = (H_2 + D_0)_{\text{diag}} \\[2mm] U_0(e) & = (H_2 + D_0)_{\text{non diag } (e2)} \\[2mm] U_0(o) & = (H_2 + D_0)_{\text{non diag } (o2)} , \end{cases}$$

$U_0(e)$ creates or annihilates an even multiple of 2 mesons. $U(o)$ creates or annihilates an odd multiple of 2 mesons.

By induction from (2.65) we define the following quantities:

$$(2.66) \qquad \begin{cases} v_n & = \big(U_{n-1}(e) + D_n\big)_{\text{diag}} , \\[2mm] U_n(e) & = \big(U_{n-1}(e) + D_n\big)_{\text{non diag } (e2^{n+1})} , \\[2mm] U_n(o) & = \big(U_{n-1}(e) + D_n\big)_{\text{non diag } (o2^{n+1})} , \\[2mm] \mathscr{V}_n & = \mathscr{V}_{n-1} + v_n = \sum_{i=0}^{n} v_i , \end{cases}$$

with

$$(2.67) \qquad D_n = U_{n-1}(o) \, \frac{1}{A - \mathscr{V}_{n-1} - U_{n-1}(e)} \, U_{n-1}(o) ,$$

$U_n(e)$ creates or annihilates an even multiple of 2^{n+1} mesons, $U_n(o)$ creates or annihilates an odd multiple of 2^{n+1} mesons.

At this point it can be shown that, if $\Omega_s^{(n)}$ is of the form

$$(2.68) \qquad \Omega_s^{(n)} = \left[1 + \frac{1}{A - \mathscr{V}_n - U_n(e) - U_n(o)} \big(\mathscr{V}_n + U_n(e) + U_n(o)\big) \right]_{\text{diag}} .$$

the $\Omega_s^{(n+1)}$ is of the form

$$(2.69) \qquad \Omega_s^{(n+1)} = \left[1 + \frac{1}{A - \mathscr{V}_{n+1} - U_{n+1}(e) - U_{n+1}(o)} \big(\mathscr{V}_{n+1} + U_{n+1}(e) + U_{n+1}(o)\big) \right]_{\text{diag}} ,$$

where $U_{n+1}(e)$, $U_{n+1}(o)$, $\mathscr{V}_{n+1}$ have an exactly analogous definition as (2.66),

(2.67). Proceeding in this way for n large enough, if the matrix elements of U become negligible, so that the series $\sum_{i=0}^{\infty} v_i$ converges, one obtains

$$(2.70) \qquad \Omega_s = 1 + \frac{1}{A - \mathscr{V}} \mathscr{V},$$

where $\mathscr{V}$ is the sum of the series $\sum_{i=0}^{\infty} v_i$.

The quantity $\mathscr{V}$ is diagonal in meson occupation number and is assumed by BRUECKNER and WATSON as the definition of the two-nucleon interaction operator.

The method here displayed can be applied to the case of the Hamiltonian (1.23)

$$H'_{\mathrm{ps}} = \frac{g_1}{2K} \int \varrho(\boldsymbol{x})(\boldsymbol{\sigma} \cdot \nabla)(\underset{\rightarrow}{\tau} \cdot \underset{\rightarrow}{\boldsymbol{\Phi}}) \, \mathrm{d}\boldsymbol{x} + \lambda \, \frac{g_1^2}{2K\hbar c} \int \varrho(\boldsymbol{x})(\underset{\rightarrow}{\tau} \cdot \underset{\rightarrow}{\boldsymbol{\Phi}})^2 \, \mathrm{d}\boldsymbol{x},$$

(the case of a Hamiltonian which is non-diagonal in nucleon occupation number, for instance the Hamiltonian (1.17)

$$H'_{\mathrm{ps}} = i g_1 \int \overline{\psi} \gamma_5 \underset{\rightarrow}{\tau} \psi \underset{\rightarrow}{\boldsymbol{\Phi}} \mathrm{d}\boldsymbol{x}$$

requires a few small changes). Denoting by H_p the meson-nucleon p-wave interaction term (which is linear in the meson field variables), by H_s the meson-nucleon s-wave interaction term (which is quadratic in the meson field variables), the results are

$$(2.71) \qquad v_0 = \left[H_s + H_p(W - H_0 - H_s)^{-1} H_p \right]_{\mathrm{diag}},$$

$$(2.72) \qquad v_1 = \left[U_0(o)\big(W - H_0 - v_0 - U_0(e)\big)^{-1} U_0(o) \right]_{\mathrm{diag}}.$$

Restricting ourselves to the contribution to the potential arising from the exchange of at most two mesons, these two expressions are sufficient; regarding H_s as a perturbation and expanding in powers of H_s the result is

$$(2.73) \qquad \mathscr{V}_1 = H_s^{(--)} \frac{1}{A} H_s^{(++)} + H_p^{(-)} \frac{1}{A} H_s^{(+-)} \frac{1}{A} H_p^{(+)} +$$

$$+ H_p^{(-)} \frac{1}{A} H_p^{(-)} \frac{1}{A} H_s^{(++)} + H_p^{(-)} \left[A - H_p^{(-)} \frac{1}{A} H_p^{(+)} \right]^{-1} H_p^{(+)}.$$

The meaning of $H_s^{(--)}$, $H_s^{(++)}$, $H_s^{(+-)}$, $H_p^{(+)}$, etc., is obvious. The last term exhibits the effects of multiple scattering. This term may in fact be written

in the form

$$H_p^{(-)} \omega \frac{1}{A} H_p^{(+)} ,$$

where ω is a solution of the equation

$$\omega = 1 + \frac{1}{A} u\omega ,$$

in which

$$u = H_p^{(-)} \frac{1}{A} H_p^{(+)}$$

acts as a « potential » for the scattering of the meson emitted by $H_p^{(+)}$ and absorbed by $H_p^{(-)}$.

The evaluation of the multiple scattering effects is the main advantage of the Brueckner and Watson method; according to the evaluation of B.W. these effects are negligible for $x > 0.5$ and therefore the last term may be written with a good approximation

$$v_a = H_p^{(-)} \frac{1}{A} H_p^{(+)} + H_p^{(-)} \frac{1}{A} H_p^{(-)} \frac{1}{A} H_p^{(+)} \frac{1}{A} H_p^{(+)} .$$

In the static limit in which $A \to - H_M$ the expression for the one-meson and two-meson exchange potential respectively are

(2.74) $$V_2 = \mu c^2 \left(\frac{\mu}{2M}\right)^2 \frac{g_1^2}{4\pi\hbar c} (\vec{\tau}^{(1)} \cdot \vec{\tau}^{(2)}) \cdot$$

$$\cdot \left[\frac{1}{3} (\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)}) + \frac{1}{3} S_{12}\left(1 + \frac{3}{x} + \frac{3}{x^2}\right)\right] \frac{\exp[-x]}{x} ,$$

(2.75) $$V_4 = - \mu c^2 \lambda^2 \left(\frac{\mu}{2M}\right)^2 \left(\frac{g_1^2}{4\pi\hbar c}\right)^2 \frac{6}{\pi} \frac{1}{x^2} K_1(2x) +$$

$$+ \mu c^2 \lambda \left(\frac{\mu}{2M}\right)^3 \left(\frac{g_1^2}{4\pi\hbar c}\right)^2 6 \left(\frac{1+x}{x^2}\right)^2 \exp[-2x] -$$

$$- \mu c^2 \left(\frac{\mu}{2M}\right)^4 \left(\frac{g_1^2}{4\pi\hbar c}\right)^2 \{R_1(x) + (\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)}) R_2(x) + S_{12} R_3(x)\} ,$$

(2.75a) $$R_1(x) \cdot \frac{2}{\pi} \left\{ (\vec{\tau}^{(1)} \cdot \vec{\tau}^{(2)}) \left[\left(\frac{12}{x^2} + \frac{23}{x^4}\right) K_1(2x) + \left(\frac{4}{x} + \frac{23}{x^3}\right) K_0(2x)\right] + \right.$$

$$\left. + (3 - 2(\vec{\tau}^{(1)} \cdot \vec{\tau}^{(2)})) \left[\left(\frac{1}{x^2} + \frac{4}{x^3} + \frac{4}{x^4}\right) K_1(x) + \left(\frac{1}{x} + \frac{2}{x^2} + \frac{2}{x^3}\right) K_0(x)\right] \exp[-x]\right\} ,$$

$$(2.75b) \qquad R_2(x) = \frac{2}{\pi} \left\{ -\left[\left(\frac{8}{x^2} + \frac{12}{x^4} \right) K_1(2x) + \frac{12}{x^3} K_0(2x) \right] + \right.$$

$$\left. + \frac{2}{3} \left(3 - 2(\underset{\rightarrow}{\tau}^{(1)} \cdot \underset{\rightarrow}{\tau}^{(2)}) \right) \left[\left(\frac{1}{x^2} + \frac{2}{x^3} + \frac{2}{x^4} \right) K_1(x) + \left(\frac{1}{x^2} + \frac{1}{x^3} \right) K_0(x) \right] \exp\left[-x \right] \right\} ,$$

$$(2.75c) \qquad R_3(x) = \frac{2}{\pi} \left\{ \left[\left(\frac{4}{x^2} + \frac{15}{x^4} \right) K_1(2x) + \frac{12}{x^3} K_0(2x) \right] - \right.$$

$$\left. - \frac{1}{3} \left(3 - 2(\underset{\rightarrow}{\tau}^{(1)} \cdot \underset{\rightarrow}{\tau}^{(2)}) \right) \left[\left(\frac{1}{x^2} + \frac{5}{x^3} + \frac{5}{x^4} \right) K_1(x) + \left(\frac{1}{x^2} + \frac{1}{x^3} \right) K_0(x) \right] \exp\left[-x \right] \right\} .$$

It must be noticed that, as can be seen from eq. (2.70), the interaction operator $\mathscr{V}$ of B.W. acts as a « potential » in a Schrödinger-like equation

$$(W - H_0) P | \Phi \rangle = \mathscr{V}(W) P | \Phi \rangle$$

for the two bare nucleons (*) and depends on the total energy W of the system.

To the procedure of B.W. therefore the same objections must be moved as those stated when discussing the Tamm-Dancoff method (**).

GARTENHAUS ([60]) obtained expressions for the potential similar to those of B.W. using pv coupling and a fixed and extended source, *i.e.* the Hamiltonian (1.22). The expressions of the Gartenhaus potential are in agreement with those of B.W. for $\lambda = 0$ except for the factor $v(k)$, the Fourier transform of ϱ, which is chosen,

$$v(k) = \exp\left[-\frac{k^2}{2k_{max}^2} \right] .$$

2ˑ4. Method of the Bethe-Salpeter equation ([28,61]). – This method is based on the relativistic two-body equation derived in the framework of field theory by Bethe and Salpeter ([62]). The problem lies in the transition from this equation to a three-dimensional equation, which may be interpreted as a Schrödinger equation for the two nucleons. This reduction, which involves a tran-

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(*) This is a consequence of (2.58) where $|\chi\rangle$ is an eigenstate of the free Hamiltonian and of the definition of Ω_s.

(**) Its actual equivalence with the Tamm-Dancoff method has been proved by KLEIN ([59]).

([59]) A. KLEIN: *Phys. Rev.*, **91**, 1285 (1953); **94**, 195 (1954).

([60]) S. GARTENHAUS: *Phys. Rev.*, **100**, 900 (1955).

([61]) A. KLEIN: *Phys. Rev.*, **90**, 1101 (1953); **94**, 1052 (1954). In the following two articles the calculation of higher order potentials and the question of convergence are discussed: A. KLEIN: *Phys. Rev.*, **91**, 740 (1953); **92**, 1017 (1953).

([62]) E. E. SALPETER and H. A. BETHE: *Phys. Rev.*, **84**, 1232 (1951); E. E. SALPETER: *Phys. Rev.*, **87**, 328 (1952).

sition to a one-time equation and, subsequently, a transition to a four-component wave function, has been examined in detail by MACKE ([63]).

Let

$$(2.76) \qquad \Psi = K\Psi$$

be the symbolic form for the B.S. equation for bound states, where $\Psi = \Psi_{\alpha^{(1)}\alpha^{(2)}}$ is a 16-component spinor and K is an operator which must be built up with the Feynman graphs, according to the well-known rules.

Let us introduce a one-time wave function $\psi = \psi_{\alpha^{(1)}\alpha^{(2)}}$ and let us define an operator J which, when applied to Ψ yields ψ:

$$(2.77) \qquad \psi = J\Psi$$

(the effect of J is therefore to equate the two times $t^{(1)}$, $t^{(2)}$ of Ψ).

For ψ an equation of the type

$$(2.78) \qquad \psi = g\psi$$

analogous to (2.76) will hold.

The eq. (2.76), (2.77), (2.78) may be written in terms of relative co-ordinates or relative momenta.

In the former case

$$\Psi = \Psi_{\alpha^{(1)}\alpha^{(2)}}(x_\mu) \,,$$

$$\psi = \psi_{\alpha^{(1)}\alpha^{(2)}}(\boldsymbol{x}, 0) \,.$$

In the latter (assuming $\boldsymbol{P} = \boldsymbol{p}^{(1)} + \boldsymbol{p}^{(2)} = 0$)

$$\Psi = \Psi_{\alpha^{(1)}\alpha^{(2)}}(p_\mu, W) \,,$$

$$\psi = \psi_{\alpha^{(1)}\alpha^{(2)}}(\boldsymbol{p}, W) \,.$$

A representation for J easily obtains in momentum space. By applying the operator J to both sides of the equation

$$(2.79) \qquad \Psi(\boldsymbol{x}, x_0) = \frac{1}{2\pi i \hbar} \int\limits_{-\infty}^{\infty} \exp\left[-i\frac{p_0}{\hbar} x_0\right] \Psi(\boldsymbol{x}, p_0) \, \mathrm{d}p_0$$

one gets

$$(2.80) \qquad \psi(\boldsymbol{x}) = \frac{1}{2\pi i \hbar} \int\limits_{-\infty}^{\infty} \Psi(\boldsymbol{x}, p_0) \, \mathrm{d}p_0 ;$$

([63]) W. MACKE: *Zeits. Naturforsch.*, **8** a, 599, 615 (1953).

hence

$$(2.81) \qquad J \to \frac{1}{2\pi i \hbar} \int\limits_{-\infty}^{\infty} \mathrm{d}p_0 \ .$$

Any summable function of p_0 is a right inverse of J, but is not in general a left inverse. Let in fact $f(p_0)$ be a function of p_0 such that

$$\frac{1}{2\pi i \hbar} \int\limits_{-\infty}^{\infty} f(p_0)\, \mathrm{d}p_0 = 1 \ ,$$

then

$$J f(p_0)\, \psi(\boldsymbol{x}) = \frac{1}{2\pi i \hbar} \int\limits_{-\infty}^{\infty} f(p_0)\, \psi(\boldsymbol{x})\, \mathrm{d}p_0 = \psi(\boldsymbol{x}) \ ,$$

while

$$f(p_0) J \Psi(\boldsymbol{x}, p_0) = f(p_0)\, \psi(\boldsymbol{x}) \neq \Psi(\boldsymbol{x}, p_0) \ .$$

Therefore the reduction of the B.S. equation to a three-dimensional equation is affected by the non-uniqueness of the right inverse of J. Nevertheless, in the case of the static interaction, $i.e.$ when retardation effects are neglected in the transmission of the interaction, the right inverse of J, J_0^{-1}, turns out to be uniquely determined as that operator which transforms ψ_{++} into Ψ_{++} (the meaning of ψ_{++} and Ψ_{++} will appear clear presently).

The result is

$$(2.82) \qquad J_0^{-1} = \frac{\hbar c\big(E^{(1)}(\boldsymbol{p}) + E^{(2)}(\boldsymbol{p}) - W\big)}{i\big(E^{(1)}(\boldsymbol{p}) - \tfrac{1}{2} W - cp_0\big)\big(E^{(2)}(\boldsymbol{p}) - \tfrac{1}{2} W + cp_0\big)} \ .$$

Expression (2.82) may be taken as a reasonable Ansatz for J_0^{-1}. Once J_0^{-1} is determined by this Ansatz, the interaction operator in the equal-time equation may be formally derived. The following equation for K is obtained

$$(2.83) \qquad K = K J_0^{-1} J + \varDelta \ ,$$

thus

$$(2.84) \qquad \Psi = K J_0^{-1} \psi + \varDelta \Psi \ ,$$

from which

$$(2.85) \qquad \Psi = (1 - \varDelta)^{-1} K J_0^{-1} \psi \ ;$$

hence

$$(2.86) \qquad \psi = J (1 - \varDelta)^{-1} K J_0^{-1} \psi$$

and the operator g of (2.78) is therefore given by

$$(2.87) \qquad\qquad g = J(1 - \Lambda)^{-1} K J_0^{-1} .$$

For the further reduction to a four component equation, eq. (2.78) is first transformed from the spinor indices $\alpha^{(1)}\alpha^{(2)}$ to the indices $\varepsilon^{(1)}\varepsilon^{(2)}$, which refer to free-particle energy and spin eigenstates, using the normalized free-particle solutions of the Dirac equation $u_{\lambda\varepsilon}(\boldsymbol{p}) \equiv u$ and $u^*_{\lambda\varepsilon}(\boldsymbol{p}) = u^\dagger$ as unitary matrices.

With the transformation

$$\psi = u^{(1)} u^{(2)} \psi' , \qquad g' = u^{(1)\dagger} u^{(2)\dagger} g u^{(1)} u^{(2)}$$

equation (2.78) becomes

$$(2.88) \qquad\qquad \psi' = g' \psi' ,$$

where

$$\psi' = \psi'_{\varepsilon^{(1)}\varepsilon^{(2)}} .$$

Denoting by ψ'_{++} the eigenfunction corresponding to positive energy states of both particles, ψ'_{+-} the eigenfunction corresponding to a positive energy state of one particle and a negative energy state of the other, etc. (each of which in turn has four components corresponding to the four spin states of the two particles), the eq. (2.88) breaks into a set of four simultaneous equations:

$$(2.89) \qquad \begin{cases} \psi'_{++} = g'_{++,++} \psi'_{++} + \sum_\eta g'_{++,\eta} \psi'_\eta , \\[2mm] \psi'_\eta = g'_{\eta,++} \psi'_{++} + \sum_{\eta'} g'_{\eta,\eta'} \psi'_{\eta'} , \end{cases}$$

(with an obvious meaning of η).

By successively inserting ψ'_η from the second equation into the first, an equation for ψ'_{++} is obtained

$$(2.90) \qquad\qquad \psi'_{++} = f \psi'_{++} ,$$

with

$$(2.91) \qquad f = g'_{++,++} + \sum_\eta g'_{++,\eta} g'_{\eta,++} + \sum_{\eta\eta'} g'_{++,\eta} g'_{\eta,\eta'} g'_{\eta',++} + \dots .$$

Using the appropriate well-known analytical forms of the graphs, f is found

to have the following form

$$(2.92) \qquad f = \left(2\,E(\boldsymbol{p}) - W\right)^{-1}\mathscr{V}\,,$$

with

$$E(\boldsymbol{p}) = c\,\sqrt{c^2\,M^2 + \boldsymbol{p}^2}\,.$$

Eq. (2.90) thus takes on the form of a Schrödinger-like equation in momentum space:

$$(2.93) \qquad \left(2E(\boldsymbol{p}) - W\right)\psi'_{++}(\boldsymbol{p}) = \frac{1}{\hbar^3}\int\mathscr{V}\left(\boldsymbol{p},\boldsymbol{p}'\,;\,W\right)\psi'_{++}(\boldsymbol{p}')\,\mathrm{d}\boldsymbol{p}'\,.$$

The operator f (and therefore $\mathscr{V}$) is built up by summing the contributions from all the permutations of points of all Feynman graphs, omitting all « product graphs » (*), *i.e.*

$$(2.94) \qquad f = \sum_{F,\pi}' f_{F,\pi}\,,$$

where $f_{F,\pi}$ is the contribution from a Feynman graph with a given topological structure and the points ordered in a given succession π; the prime in the summation symbol stands for the exclusion of « product graphs ».

The potential V therefore has the form of a power series expansion in the coupling constant. By rearranging this expansion so as to have a sum on renormalized Feynman graphs, one obtains an expansion in the number of exchanged mesons. (This assumes that the theory is renormalizable and the sum is convergent.)

The one-meson and two-meson exchange potential is obtained from the second and fourth order renormalized graphs respectively.

The calculation of this potential, at the static limit, and its expression in terms of experimental data on pion-nucleon scattering (using the dispersion relations of fixed source meson theory for p-wave and s-wave meson scattering, along the lines established by MIYAZAWA) has been outlined by KLEIN and McCORMICK [8,64].

We quote here an approximate evaluation of the static two-meson exchange potential (the one-meson exchange potential has the now familiar form) done by K.McC. for the purpose of exhibiting the order of magnitude of the « corrections » to the B.W. potential:

$$(2.95) \qquad V_4 = V_{4p} + V_{4p-s} + V_{4s}\,,$$

(*) Given two graphs f_1 and f_2 we mean by « product graphs » all the graphs $f_1^2\cdot f_2^2\ldots,f_1f_2,f_1^nf_2^m\ldots$ which may obtained by matrix multiplication of the graphs f_1 and f_2.

[64] A. KLEIN and B. H. McCORMICK: *Progr. Theor. Phys.*, **20**, 876 (1958).

$$(2.95a) \quad V_{1p} = - \mu c^2 \left(\frac{\mu}{2M}\right)^4 \left(\frac{g_1^2}{4\pi\hbar c}\right)^2 \{R_1(x) + (\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)})R_2(x) + S_{12}R_3(x)\} -$$

$$- 3\lambda_1^2 \mu c^2 \left(\frac{\mu}{2M}\right)^6 \left(\frac{g_1^2}{4\pi\hbar c}\right)^2 \frac{2}{\pi}\left[\left(\frac{43}{2x^6} + \frac{63}{4x^4} + \frac{1}{x^2}\right) K_1(2x) + \left(\frac{43}{2x^5} - \frac{5}{x^3}\right) K_0(2x)\right] +$$

$$+ \lambda_2^2 \mu c^2 \left(\frac{\mu}{2M}\right)^6 \left(\frac{g_1^2}{4\pi\hbar c}\right)^2 \frac{2}{\pi} (\vec{\tau}^{(1)}\cdot\vec{\tau}^{(2)}) \cdot$$

$$\cdot \left\{\frac{4}{3} (\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)}) \left[\left(\frac{10}{x^6} + \frac{7}{x^4}\right) K_1(2x) + \left(\frac{10}{x^5} + \frac{2}{x^3}\right) K_0(2x)\right] -\right.$$

$$\left. - \frac{2}{3} S_{12} \left[\left(\frac{35}{2x^6} + \frac{10}{x^4}\right) K_1(2x) + \left(\frac{35}{2x^5} + \frac{2}{x^3}\right) K_0(2x)\right]\right\} .$$

$$- 6\lambda_1 \mu c^2 \left(\frac{\mu}{2M}\right)^5 \left(\frac{g_1^2}{4\pi\hbar c}\right)^2 \frac{\exp[-2x]}{x^2}\left[\frac{6}{x^4} + \frac{12}{x^3} + \frac{10}{x^2} + \frac{4}{x} + 1\right] +$$

$$+ \lambda_2 \mu c^2 \left(\frac{\mu}{2M}\right)^5 \left(\frac{g_1^2}{4\pi\hbar c}\right)^2 \left\{\frac{8}{3} (\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)})(\vec{\tau}^{(1)}\cdot\vec{\tau}^{(2)}) \frac{\exp[-2x]}{x}\left[\frac{3}{x^4} + \frac{6}{x^3} + \frac{5}{x^2} + \frac{2}{x}\right] -\right.$$

$$\left. - \frac{4}{3} (\vec{\tau}^{(1)}\cdot\vec{\tau}^{(2)})S_{12} \frac{\exp[-2x]}{x^2}\left[\frac{6}{x^4} + \frac{12}{x^3} + \frac{8}{x^2} + \frac{2}{x}\right]\right\} ,$$

$$(2.95b) \quad V_{4s-p} = 6\varrho_1 \mu c^2 \left(\frac{\mu}{2M}\right)^3 \left(\frac{g_1^2}{4\pi\hbar c}\right)^2 \left(\frac{1+x}{x^2}\right)^2 \exp[-2x] +$$

$$+ \varrho_2 \mu c^2 \left(\frac{\mu}{2M}\right)^4 \left(\frac{g_1^2}{4\pi\hbar c}\right)^2 \frac{8}{\pi} (\vec{\tau}^{(1)}\cdot\vec{\tau}^{(2)}) \left[\left(\frac{5}{2x^4} + \frac{1}{x^2}\right) K_1(2x) + \frac{5}{2x^3} K_0(2x)\right] +$$

$$+ \lambda_1 \varrho_1 \mu c^2 \left(\frac{\mu}{2M}\right)^4 \left(\frac{g_1^2}{4\pi\hbar c}\right)^2 \frac{12}{\pi} \left[\left(\frac{5}{2x^4} + \frac{1}{x^2}\right) K_1(2x) + \frac{5}{2x^3} K_0(2x)\right],$$

$$(2.95c) \quad V_{4s} = - 3\varrho_1^2 \mu c^2 \left(\frac{\mu}{2M}\right)^2 \left(\frac{g_1^2}{4\pi\hbar c}\right)^2 \frac{2}{\pi} \frac{K_1(2x)}{x^2} +$$

$$+ \varrho_2^2 \mu c^2 \left(\frac{\mu}{2M}\right)^4 \left(\frac{g_1^2}{4\pi\hbar c}\right)^2 \frac{2}{\pi} (\vec{\tau}^{(1)}\cdot\vec{\tau}^{(2)}) \left[\frac{K_1(2x)}{x^4} + \frac{K_0(2x)}{x^3}\right].$$

The expressions (2.95, a, b, c) are, respectively, the contributions from p-wave, from p-wave and s-wave interference, and from s-wave; in each of them the first term is what would be obtained treating the meson-nucleon scattering amplitude in Born approximation and is in agreement with the corresponding term of the B.W. potential; λ_1 and λ_2 are parameters which measure the weight of the deviation of p-wave scattering from Born approximation; ϱ_1 and ϱ_2 are s-waves damping factors, *i.e.* they play a role similar to that of the factor λ introduced in (1.23). These parameters are determined on the basis of the analysis of experimental data of pion-nucleon scattering; using the accepted values of meson-nucleon scattering amplitudes, K.McC. adopt the following values: $\varrho_1 - 0.01$, $\varrho_2 = 0.57$, $\lambda_1 - 2.9(M/\mu)$, $\lambda_2 - 1.3(M/\mu)$.

3. – Remarks on the various theoretical potentials.

3‘1. *Static potentials.* – We turn now to a description of the behaviour of the static potentials obtained by the various authors.

The potentials are plotted in the various diagrams for triplet even states, singlet even states, triplet odd states, and singlet odd states, respectively. For the sake of comparison the diagrams of Fig. 1a), b), c), d) exhibit, together with the complete potential $V = V_2 + V_4$ of B.W. the one-pion exchange potential.

The asymptotic behaviour of the latter, as can be seen from the expression

$$V_2(\mathbf{r}) = \frac{G^2}{4\pi\hbar c}\,\mu c^2(\underset{\rightarrow}{\tau}^{(1)}\cdot\underset{\rightarrow}{\tau}^{(2)})\left[\frac{1}{3}(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)}) + \frac{1}{3}\,S_{12}\left(1 + \frac{3}{x} + \frac{3}{x^2}\right)\right]\frac{\exp[-x]}{x}\,,$$

$$G = g_2 = g_1\,\frac{\mu}{2M}\,.$$

which is consistently obtained by all authors, is of the type e^{-x} whereas the asymptotic behaviour of the two-pion exchange potential is of the type e^{-2x} as was to be expected. The behaviour of V_2 is recorded in the diagram for $G^2/4\pi\hbar c = 0.08$, a value determined from the analysis of deuteron and nucleon-nucleon scattering data (*).

From the diagrams three features must be pointed out:

1) the central part of V_2 is attractive but very weak;

2) the tensor part of V_2 has the sign favourable for the deuteron electric quadrupole moment and is sufficiently strong;

3) the corrections which the complete potential $V = V_2 + V_4$ brings to the potential V_2 are relevant for $0.5 < x < 1$ and negligible for $x > 1.5$ (as was to be expected), but are not wholly negligible for $1 < x < 1.5$ (**) (***).

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(*) This value is in good agreement with that obtained from the analysis of meson-nucleon scattering. The experimental verification of the one-pion exchange potential will be the subject of the second part of our article.

(**) The validity of these considerations is not limited to the B.W. potential, which is here taken only for argument's sake.

(***) A preliminary evaluation of the three-pion exchange potential for the purpose of estimating the nucleon separation at which this becomes comparable with the one-plus-two-pion exchange potential, has been done by MACHIDA and SENBA [65]. According to their calculations the three-pion exchange potential is estimated to be negligible for $x \geqslant 0.7$.

[65] S. MACHIDA and K. SENBA: *Prog. Theor. Phys.*, **13**, 389 (1955).

T.M.O. potential. The T.M.O. (2.43), (2.44) is plotted in the diagrams of Fig. 2 *a*), *b*), *c*), *d*).

In the triplet odd states, besides a strong tensor «attraction», which is suited to account for the deuteron quadrupole moment, it yields a strong central repulsion; this central repulsion is an unfavourable hint on the T.M.O. potential. It should be noted however that the modulus of the central part is less than the modulus of the tensor part for $x > 0.68$, while it is larger for $x < 0.68$, *i.e.* virtually in the phenomenological region. Therefore a treatment of deuteron and of low energy scattering in triplet even states with the T.M.O. potential for $x > 0.6$ and a suitable phenomenological adjustement for $x < 0.68$ might not be impossible. On the other hand if for the phenomenological inter-action in the inner region a reasonable «hard core» is assumed (according to the attitude most common today) the T.M.O. potential is faced by serious difficulties.

In the singlet even states there is a strong attraction, and this is a favour-able indication regarding the low-energy singlet scattering parameters. It should be remarked however that the value of the T.M.O. potential around $x = 1$ might be too small to give a satisfactory value for the singlet effective range.

In the triplet odd state there is only a weak potential in the region $x > 0.7$; this might be a favourable feature for the explanation of the observed angular distribution in high-energy n-p scattering.

B.W. potential. Its expression is given by (2.74), (2.75) and its behaviour is plotted in the diagrams of Fig. 3 *a*), *b*), *c*), *d*) for different values of λ. It can be seen that in singlet even and odd states the potential is not very sen-sitive to the value of λ, while in triplet states it is rather sensitive to the value of λ. This sensitivity would appear to be more sharp for small values of λ.

In the triplet even state the central part of the B.W. potential turns out to be radically different from the T.M.O. potential. A comparison with the T.M.O. potential can be made from the diagrams of Fig. 4 *a*), *b*), *c*), *d*) where the B.W. potential is plotted for $\lambda = 0$. It is seen that the potential for singlet even and odd states, the central part and the tensor part of triplet even states are in agreement with or not very different from the corresponding T.M.O. potentials; the central part of triplet even states exhibits a remarkable attraction instead of T.M.O.'s repulsion.

This difference has been widely debated. It stems from the term

$$\delta V_4 = \mu c^2 \left(\frac{G^2}{4\pi\hbar c}\right)^2 \frac{2}{\pi} (\vec{\tau}^{(1)} \cdot \vec{\tau}^{(2)}) \frac{\exp[-x]}{x} \left[\frac{4 + 4x + x^2}{x} K_1(x) + \right.$$

$$+ (2 + 2x + 2x^2)K_0(x) + \frac{2}{3} (\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)}) \left((1 + x)K_0(x) + \frac{2 + 2x + x^2}{x} K_1(x)\right) -$$

$$\left. - \frac{1}{3} S_{12} \left((1 + x)K_0(x) + \frac{5 + 5x + x^2}{x} K_1(x)\right)\right].$$

According to B.W. and other authors ([64,66]) this term is the first of a series of terms coming from an expansion of the non-adiabatic part of the one-pion interaction. An expansion of this kind might be expected to be not very fastly convergent; hence it might be unjustified to consider only the first term. The contribution from the whole series (without expansion) is evaluated by B.W. to be negligible for a reasonable deuteron wave function. Hence the omission of the term in question from the expression of the potential.

Other authors ([30,34,35,51,52]) object to B.W.'s point of view that this procedure fails short in the case of the neutral scalar theory, inasmuch as the exact solution, for the *fixed source*, which is known in another way ([36]), requires the inclusion of the term omitted by B.W.

According to the point of view of F.S.T. the difference arises from the different treatment of the probability operator $1/J^\dagger J$. The B.W. potential is obtained when assuming the value 1 for the probability operator, while the T.M.O. potential is obtained when expanding the probability operator up to the second order in the coupling constant.

Gartenhaus potential. The Gartenhaus potential is plotted in the diagrams of Fig. 5 a), b), c), d). In the triplet even state the tensor part is « attractive » for the one-pion exchange potential, « repulsive » for the two-pion exchange potential, about one third as large as the first. The result is an « attractive » tensor interaction which vanishes at the origin. The central part exhibits a weak attraction mainly due to the two-pion exchange; it becomes repulsive for $x \leqslant 0.6$.

In singlet even states there is a strong attraction (with a maximum of 70 MeV at $x = 0.6$, with $G^2/4\pi\hbar c = 0.08$ and $\omega_{max} = 6\mu$) almost entirely due to the two-pion exchange; the interaction is then strongly repulsive for $x \leqslant 0.4$.

In triplet odd states the central part is everywhere attractive; the tensor part is « repulsive » for $x \geqslant 0.6$, becomes « attractive » for $x \leqslant 0.6$ and tends to zero for $x \to 0$.

In singlet odd states there is a repulsion for $x \geqslant 0.75$ and an attraction for $x < 0.75$ so strong as to give a bound state $^1P_1$.

It must be remarked however that the characteristic features of the Gartenhaus potential occur inside the inner region whose phenomenological character was not recognized by Gartenhaus, claiming to extend the validity of his potential up to the origin.

F.S.T. potential. The treatment of the probability factor makes the F.S.T. potential (2.32) a form of potential intermediate between the T.M.O. and the B.W. potentials, as is also apparent from the diagrams of Fig. 6 a), b), c), d).

([66]) A. KLEIN: *Progr. Theor. Phys.*, **20**, 257 (1958).

While it does not differ very much from the T.M.O. and B.W. potential for the tensor part of triplet even and odd states and the central part of triplet odd states, remarkable differences both from the T.M.O. potential and from the B.W. potential arise in the other cases. In triplet even states the strong central repulsion of T.M.O. is greatly weakened, although it remains a repulsion; in singlet even states the attraction of T.M.O. and B.W. is notably weakened for $x > 0.6$ and actually becomes a repulsion for $x < 0.6$; in singlet odd states the repulsion of T.M.O. and B.W. changes into a strong attraction for $x < 0.8$.

K.M.O. potential.   The K.M.O. potential (Fig. 7 *a*), *b*), *c*), *d*)) is in almost complete agreement with that of T.M.O., F.S.T. and B.W. for the tensor part in triplet even states; the difference from that of T.M.O., F.S.T. and B.W. is a little more marked for the tensor part in triplet odd states; in singlet even states and in the central part of triplet odd state the attraction, of T.M.O. and B.W. is further enhanced. In singlet odd states its behaviour approaches that of F.S.T. although somewhat damped, and therefore differs sharply from that of T.M.O. and B.W. On the other hand the central part of the triplet even state comes near to that of B.W. and is therefore very different from that of T.M.O.

In the K.M.O. potential, where no pair suppression is assumed, there is also a certain contribution from the *s*-wave, which however turns out to be rather small, except for the charge singlet states.

Klein and McCormick potential.   The term corresponding to the contribution from the *s*-wave, like in the K.M.O. potential, from which this potential (2.95) does not essentially differ, gives a negligibly small contribution and is therefore omitted by KLEIN and MCCORMICK. Its behaviour once this term is neglected, is plotted in the diagrams for the singlet even and triplet even states (Fig. 8 *a*), *b*)).

For the singlet even states the strong attraction of B.W. is sharpened by the corrective terms occurring in the potential of Klein and McCormick. In triplet even states the corrective terms on one hand sharpen the central attraction, on the other they weaken the tensor « attraction » of the B.W. potential. The central part thus notably differs both from that of T.M.O. and that of K.M.O. (This is obvious since KLEIN and MCCORMICK adopt B.W.'s prescription regarding the controversial terms.) In simple terms it may be said that the this potential is to the B.W. potential what the K.M.O. potential is to the T.M.O. potential.

3˙2. *Non-adiabatic corrections and* ($L \cdot S$) *term.*   The greatest efforts have been directed to the derivation of the static nucleon-nucleon potential. Ex-

tensive ambiguities prevent at the present stage of the theory a systematic study of non-adiabatic corrections.

Experiment has recently pointed to the convenience (*) of considering velocity dependent terms: $(L \cdot S)$ ([5,67]) coupling and also, more recently, of the type

$$(\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{p})(\boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{p}) \qquad \text{and} \qquad \tfrac{1}{2}[(\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{L})(\boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{L}) + (\boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{L})(\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{L})] \,.$$

A certain number of studies have been done in order to derive from meson theory terms of this kind. Since the Thomas spin-orbit coupling term ([68]), which is of order $(\mu/M)^2$, is too small to be satisfactory, it is clear that a spin-orbit coupling term of adequate magnitude may be derived only taking into account the retardation effect in the interaction between the two nucleons. The usual way to evaluate this effect is to expand the interaction kernel in a power series of $p/Mc \approx \mu/M$.

In this way ARAKI ([69]) was able to show that from the interaction kernel up to the second order in the coupling constant no spin-orbit coupling term of order μ/M relative to the static term can be derived from pseudoscalar meson theory. Later ([70]), carrying out the calculations on the interaction kernels derived in the framework of pv coupling theory by I. SATO ([47]), he obtained a spin-orbit coupling term

$$(3.1) \quad V_{\text{s.o.}} = -\,48\mu c^2 \left(\frac{G^2}{4\pi\hbar c}\right)^2 \left(\frac{\mu}{2M}\right) \frac{\exp\left[-2x\right]}{x^6}\left[1 + \frac{3}{2}\,x^2 + 2x + \frac{1}{2}\,x^3\right] (L \cdot S) \,,$$

which is isotopic spin independent, since the terms depending on isotopic spin were evaluated to be negligible.

High energy p-p scattering experiments recently have suggested that spin-orbit coupling depends on isotopic spin ([71]). More accurate evaluations of the $(L \cdot S)$ coupling term have been performed by SHINDO and NISHIJIMA ([72]), by S. SATO ([73]) (using the F.S.T. method and the T.M.O. prescription for the probability operator), by I. SATO, K. ITABASHI and S. SATO ([74]) (using the

(*) We do not go here into the question whether velocity-dependent terms are necessary to account for nucleon-nucleon experimental data.

[67] J. L. GAMMEL and R. M. THALER: *Phys. Rev.*, **107**, 291, 1337 (1957).

[68] G. BREIT: *Phys. Rev.*, **51**, 248 (1937); **53**, 153 (1938).

[69] G. ARAKI: *Prog. Theor. Phys.*, **6**, 379 (1951).

[70] G. ARAKI: *Progr. Theor. Phys.*, **13**, 13 (1955).

[71] D. S. SIGNELL, R. ZINN and R. E. MARSHAK: *Phys. Rev. Lett.*, **1**, 416 (1958); M. H. HULL, K. D. PYATT, C. R. FISHER and G. BREIT: *Phys. Rev. Lett.*, **3**, 264 (1959).

[72] M. SHINDO and K. NISHIJIMA: *Progr. Theor. Phys.*, **13**, 103 (1955).

[73] S. SATO: *Progr. Theor. Phys.*, **13**, 457 (1955).

[74] I. SATO, K. ITABASHI and S. SATO: *Progr. Theor. Phys.*, **14**, 303 (1955).

F.S.T. method and the F.S.T. prescription for the probability operator) on the basis of pv coupling theory. As an example we record the results of S. Sato:

$$(3.2) \qquad V_{s.o.} = -\, 4\mu c^2 \left(\frac{G'^2}{4\pi\hbar c}\right)^2 \left(\frac{\mu}{2\,M}\right) \cdot$$

$$\cdot \left\{ (1+x)\,\frac{1+x+x^2}{x^6}\,\exp\left[-\,2x\right][3 + 2(\vec{\tau}^{(1)}\cdot\vec{\tau}^{(2)})] \right\} (\boldsymbol{L}\cdot\boldsymbol{S}).$$

As can be seen from (3.2), it turns out to be positive in charge singlet states and negative in charge triplet states (the spin-orbit coupling term derived by I. Sato, K. Itabashi and S. Sato on the contrary is negative for both charge states). In the shell model of nuclear structure the terms depending on $\vec{\tau}$ seem to be negligible; the $V_{s.o.}$ of S. Sato would thus have the « right sign ».

More refined calculations on the basis of the Klein and McCormick method have been done by Tzoar, Raphael and Klein ([75]). Okubo and Marshak ([3]) with a preliminary evaluation were able to show that quadratic terms of the type

$$\tfrac{1}{2}[(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{L})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{L}) + (\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{L})(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{L})] - (\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)})L^2$$

may be derived from meson theory; they follow from the expansion up to the second order in $\mu/2M$ of the interaction kernel up to the second order in G.

The most complete evaluation of the spin-orbit coupling terms up to the order $(G^2/4\pi\hbar c)(\mu/2\,M)$ and $(G^2/4\pi\hbar c)(\mu/2\,M)^2$ is, up to now, that of Sugawara and Okubo ([76]). From their results there appears a relevant difference between the ps-ps and ps-pv theories. In the former the $(\boldsymbol{L}\cdot\boldsymbol{S})$ coupling term, of order μ/M arises from the one nucleon-antinucleon pair processes alone. In the latter the main term follows from the expansion of the spin matrix elements in the one-pion exchange interaction kernel. On account of its origin in the two theories, the $(\boldsymbol{L}\cdot\boldsymbol{S})$ coupling term seems to be more firmly grounded in the ps-pv theory than in the ps-ps theory. Moreover it turns out to be generally of greater magnitude in the ps-pv theory than in the ps-ps theory.

On account of the various reasons mentioned, we must conclude that the treatment of the spin-orbit coupling terms is still inedaquate. In the first place in ps-ps theory the terms in question arise, at the lowest order in μ/M, from

([75]) N. Tzoar, R. Raphael and A. Klein: *Phys. Rev. Lett.*, **2**, 433 (1959); **3**, 145 (1959).

([76]) M. Sugawara and S. Okubo: *Phys. Rev.*, **117**, 605, 614 (1960).

the nucleon-antinucleon pair processes and are therefore sensitive to pair suppression. IWADARE [77] showed that the survival of these terms is tied to a partial pair suppression; besides the obvious case of complete suppression, they also vanish in the case of no suppression because of the cancellation between the two-pair term, previously obtained by KLEIN [78], and the one-pair term.

Secondly the series expansion in powers of p/Mc might be badly convergent, if at all. Even if the convergence of the expansion is admitted, cancellations might arise when developing up to higher orders.

Recent very suggestive ways to explain the origin of the spin-orbit term have been devised by GUPTA [79], who assumes the existence of a neutral scalar meson ϱ^0, with a mass considerably larger than the pion mass (about $2\,\mu$) and by SAKURAI [80], who assumes a resonance correlation of three pions (or a vector meson).

Conclusions.

As already pointed out the one-pion exchange potential is identical for all authors, irrespective of the method employed for its construction and of the coupling (ps or pv) used. One is therefore tempted to attach a certain degree of certainty to the one-pion exchange potential and to submit it to the test of experiment. That it should be in principle possible is made plausible by the fact that there is a peripheral region in which the one-pion exchange potential dominates the interaction; effects « localized » in this region should therefore be determined by the one-pion exchange potential. The experimental verifications of the one-pion exchange potential will be the subject of the second part of our article.

Unfortunately analogous statements cannot be made for the two-pion exchange potential which is still plagued by countless ambiguities. This is not surprising in view also of the fact that the evaluation of the various corrections, such as multiple scattering effects and radiative corrections, is still in a very preliminary stage and therefore hardly significant. Notwithstanding this various authors have attempted experimental tests of certain one-pion plus two-pion exchange potentials; we do not think that the situation as to the

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[77] J. IWADARE: *Progr. Theor. Phys.*, **13**, 189 (1955); **14**, 16 (1955).
[78] A. KLEIN: *Phys. Rev.*, **90**, 1101 (1953).
[79] S. N. GUPTA: *Phys. Rev. Lett.*, **2**, 124 (1959).
[80] J. J. SAKURAI: *The spin-orbit force and a neutral vector meson*, EFINS-60-18 (April 1960).
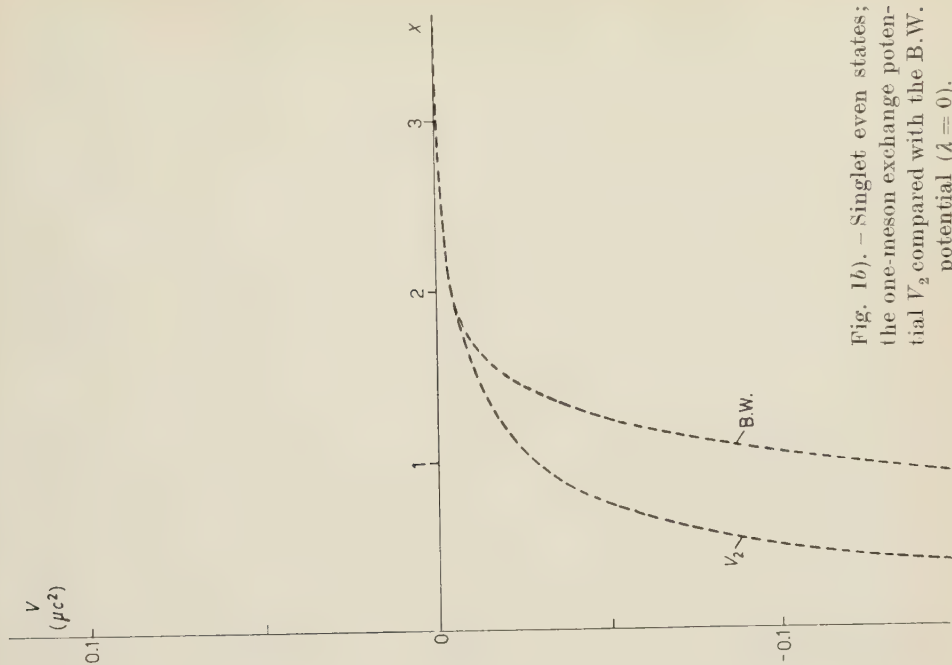
Fig. 1b). – Singlet even states; the one-meson exchange potential V_2 compared with the B.W. potential ($\lambda = 0$).

Fig. 1a). – Triplet even states; the one-meson exchange potential V_2 compared with the B.W. potential ($\lambda = 0$). (- - - central part, ——— tensor part).

Fig. 1d). – Singlet odd states: the one-meson exchange potential V_2 compared with the B.W. potential ($\lambda = 0$).
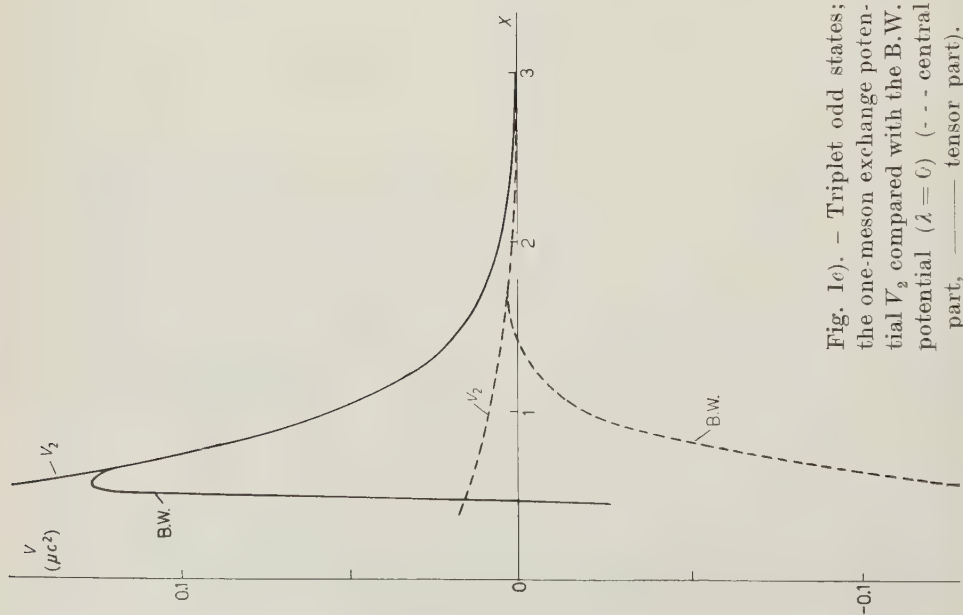
Fig. 1c). – Triplet odd states; the one-meson exchange potential V_2 compared with the B.W. potential ($\lambda = 0$) (- - - central part, ——— tensor part).
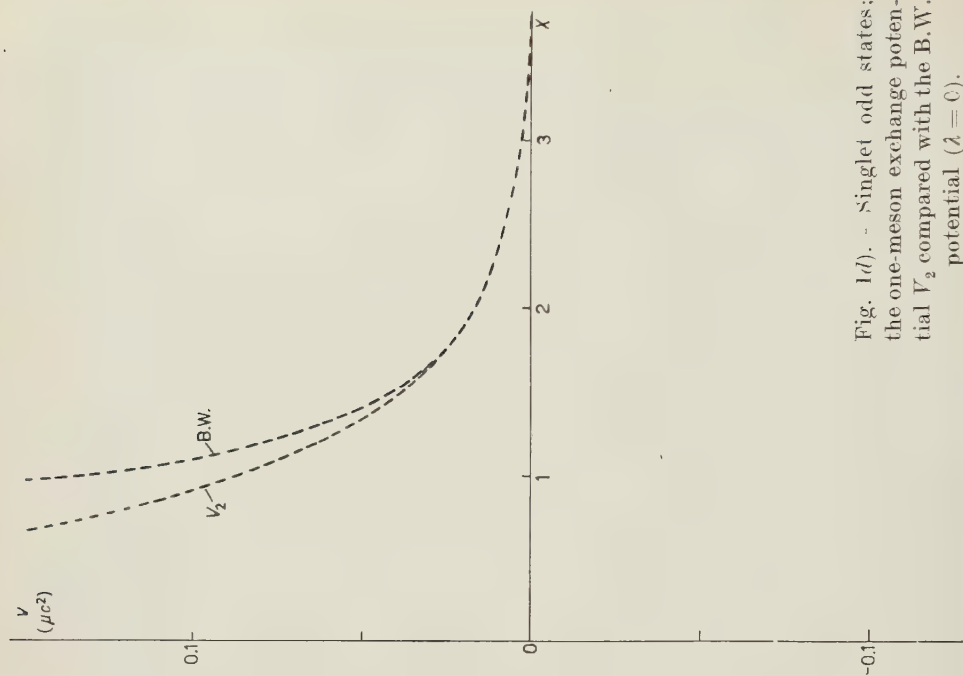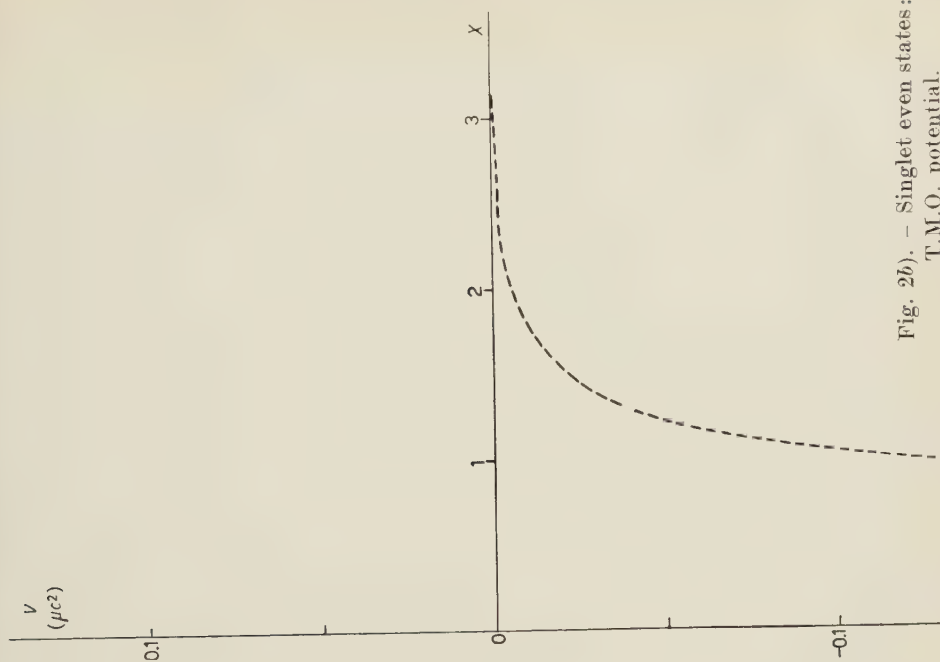
Fig. 2b). – Singlet even states: T.M.O. potential.

Fig. 2a). – Triplet even states: T.M.O. potential (- - - central part, ——— tensor part).

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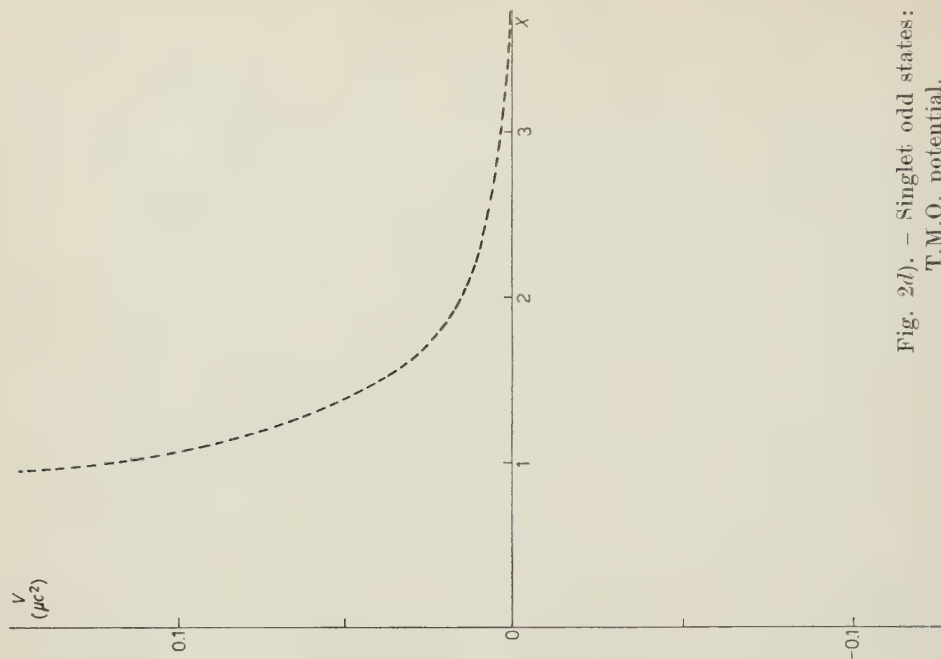
Fig. 2d). – Singlet odd states: T.M.O. potential.
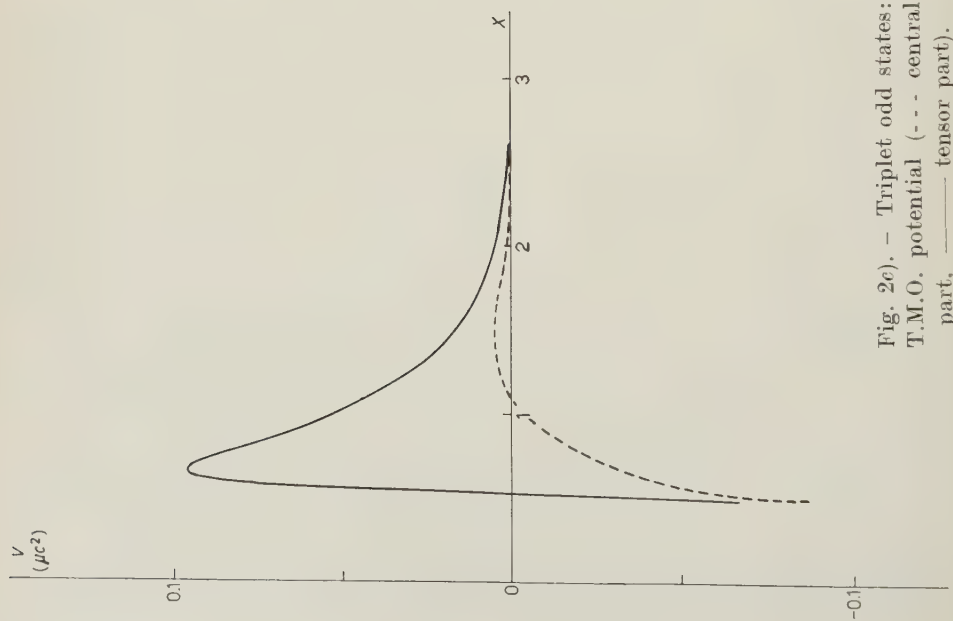
Fig. 2e). – Triplet odd states: T.M.O. potential (- - - central part, ———— tensor part).
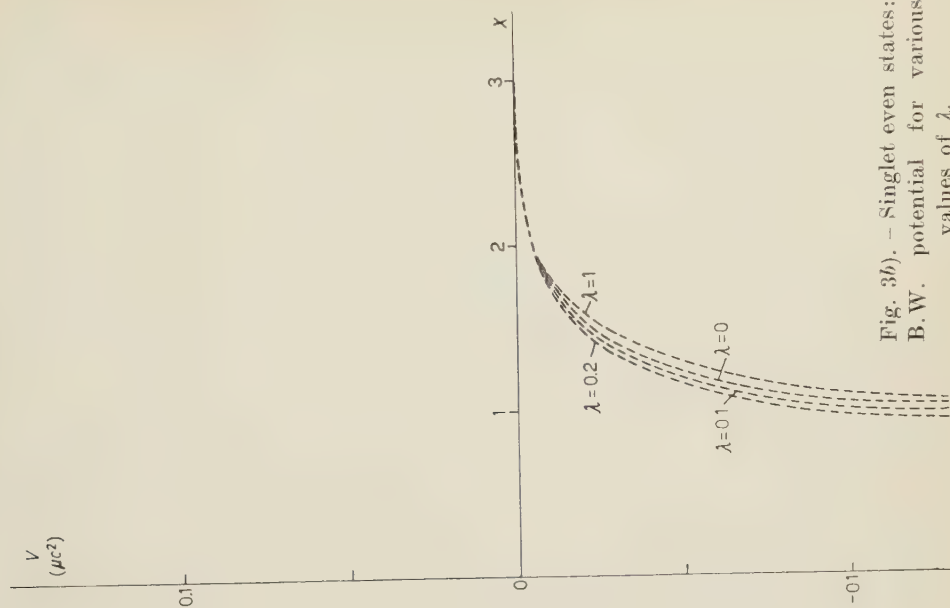
Fig. 3b). – Singlet even states: B.W. potential for various values of λ.

Fig. 3a). – Triplet even states: B.W. potential for various values of λ (- - - central part. —— tensor part)

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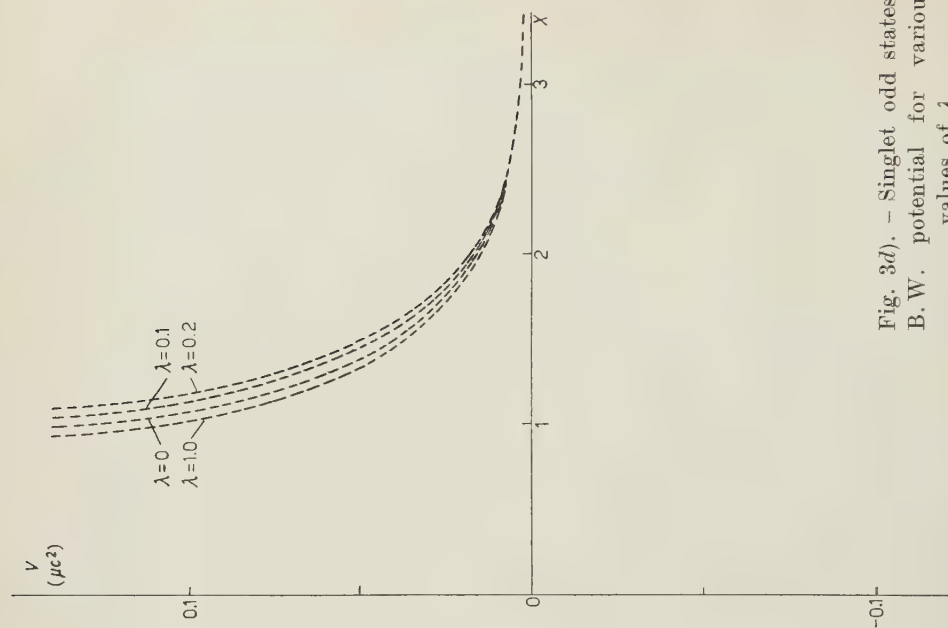
Fig. 3d). – Singlet odd states: B.W. potential for various values of λ.
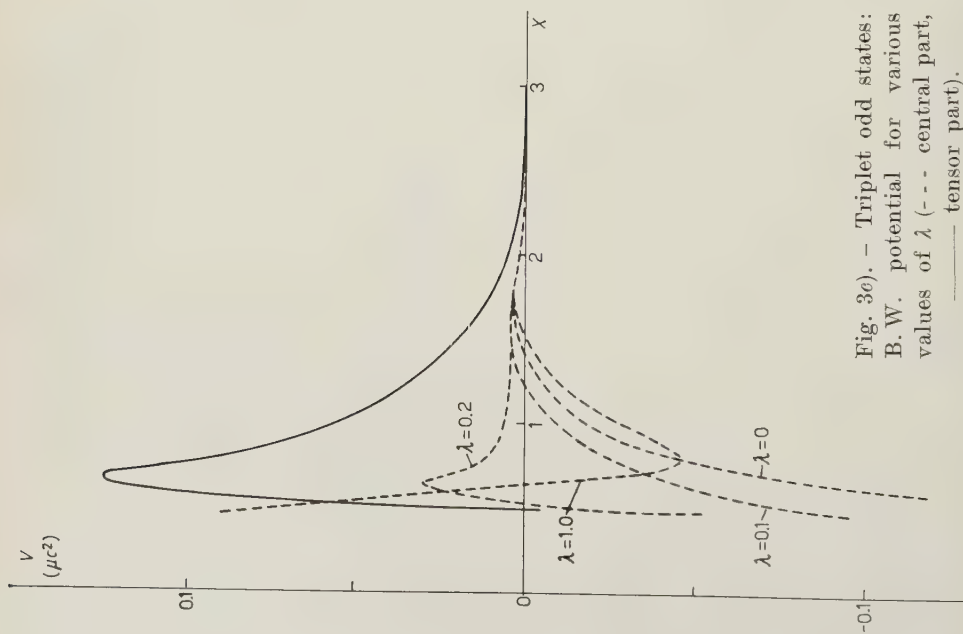
Fig. 3e). – Triplet odd states: B.W. potential for various values of λ (- - - central part, ——— tensor part).

Fig. 4b). – Singlet even states: B.W. potential ($\lambda=0$) compared with T.M.O. potential.
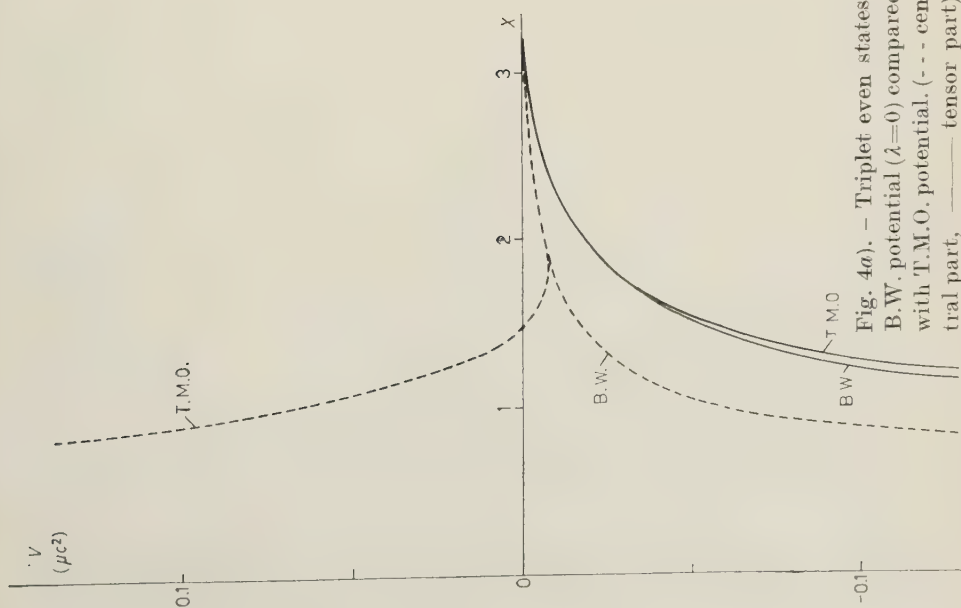
Fig. 4a). – Triplet even states: B.W. potential ($\lambda=0$) compared with T.M.O. potential. (- - - central part, ——— tensor part).
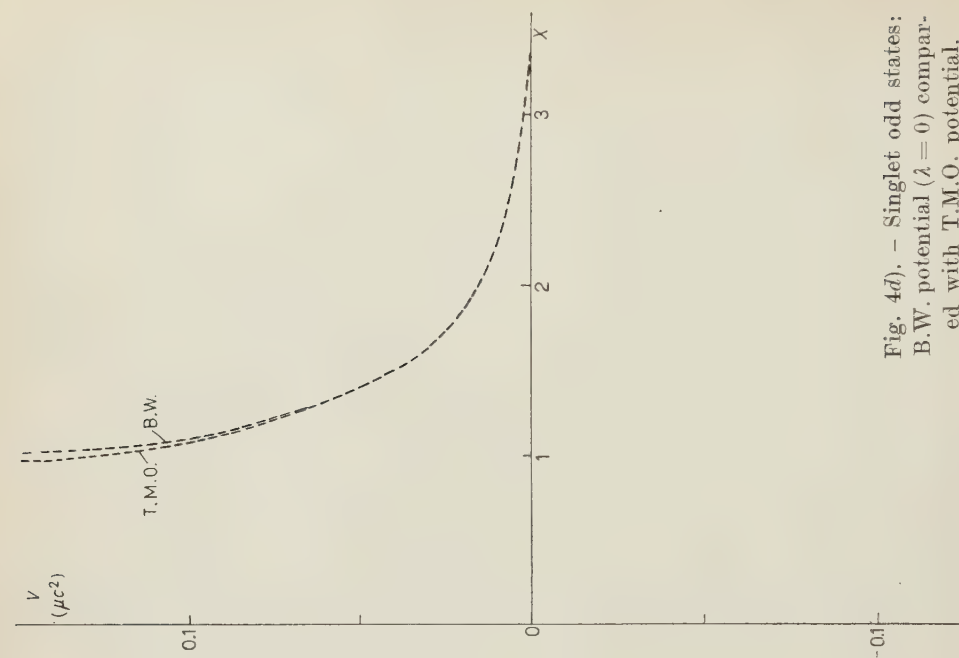
Fig. 4d). – Singlet odd states: B.W. potential ($\lambda = 0$) compared with T.M.O. potential.
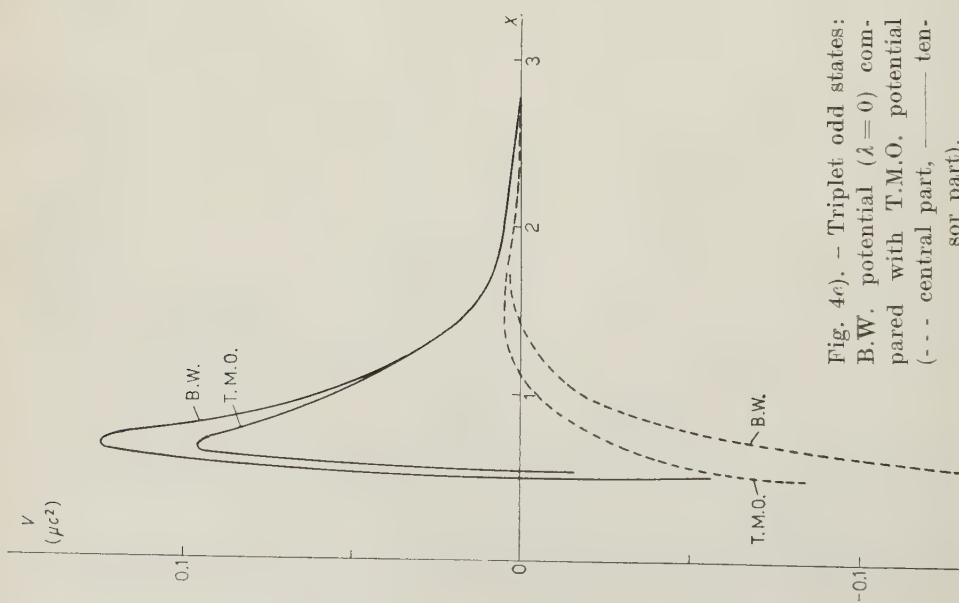
Fig. 4c). – Triplet odd states: B.W. potential ($\lambda = 0$) compared with T.M.O. potential (- - - central part, ——— tensor part).

Fig. 5b). – Singlet even states: Gartenhaus potential.
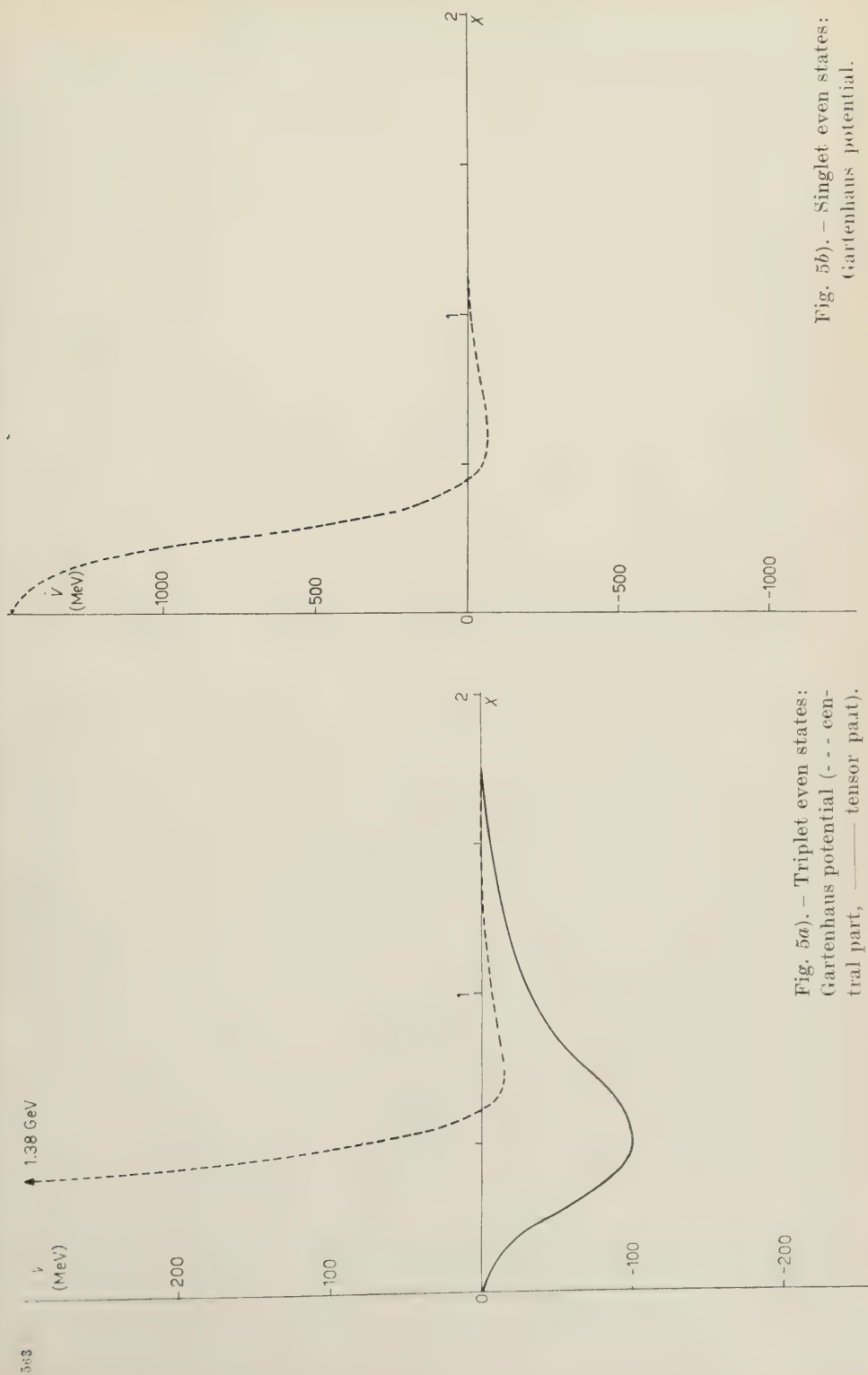
Fig. 5a). – Triplet even states: Gartenhaus potential (- - - central part, ——— tensor part).

Fig. 5d). – Singlet odd states: Gartenhaus potential.
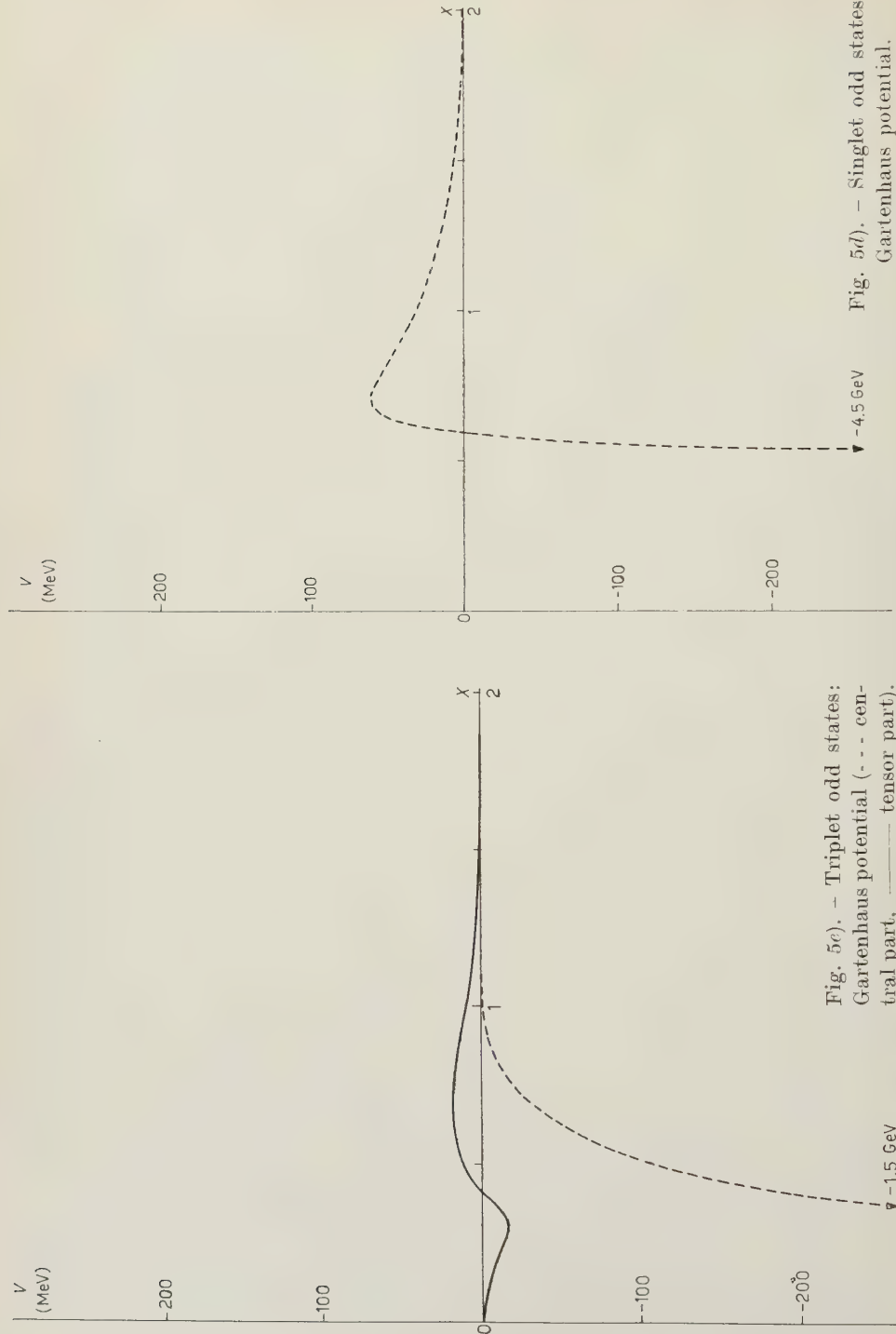
Fig. 5e). – Triplet odd states: Gartenhaus potential (- - - central part, ——— tensor part).
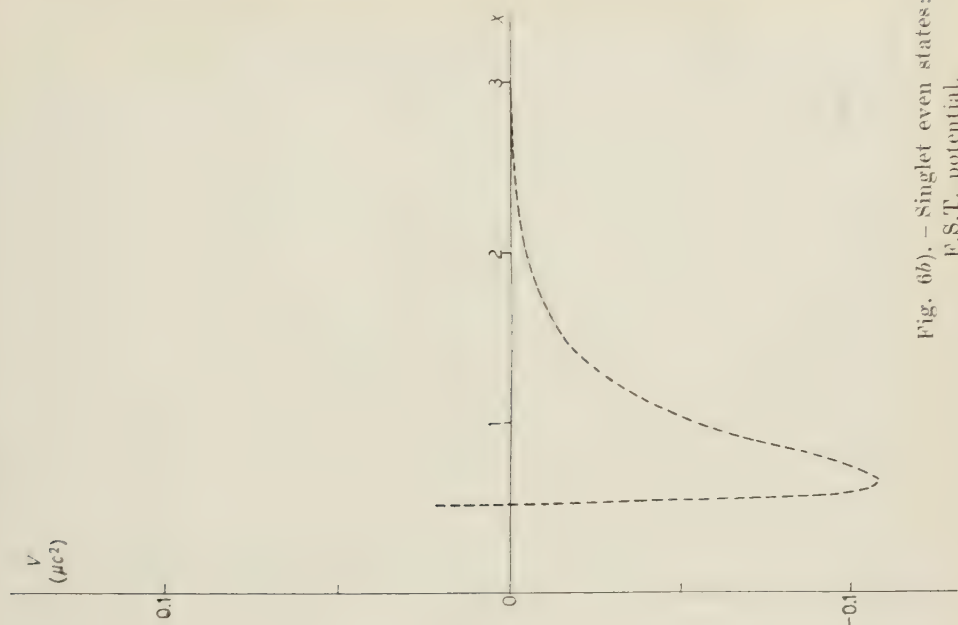
Fig. 6b). – Singlet even states: F.S.T. potential.
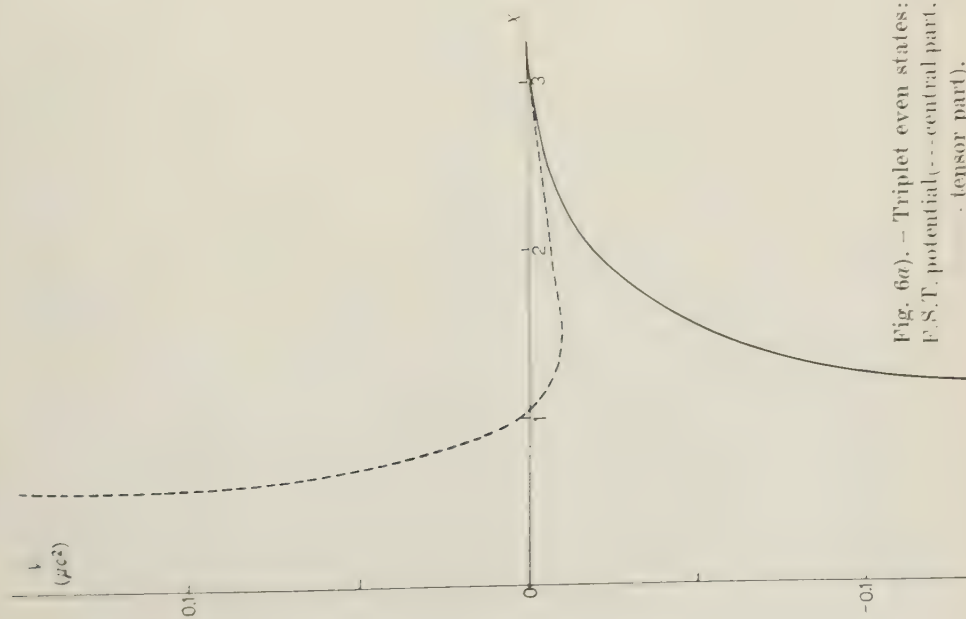
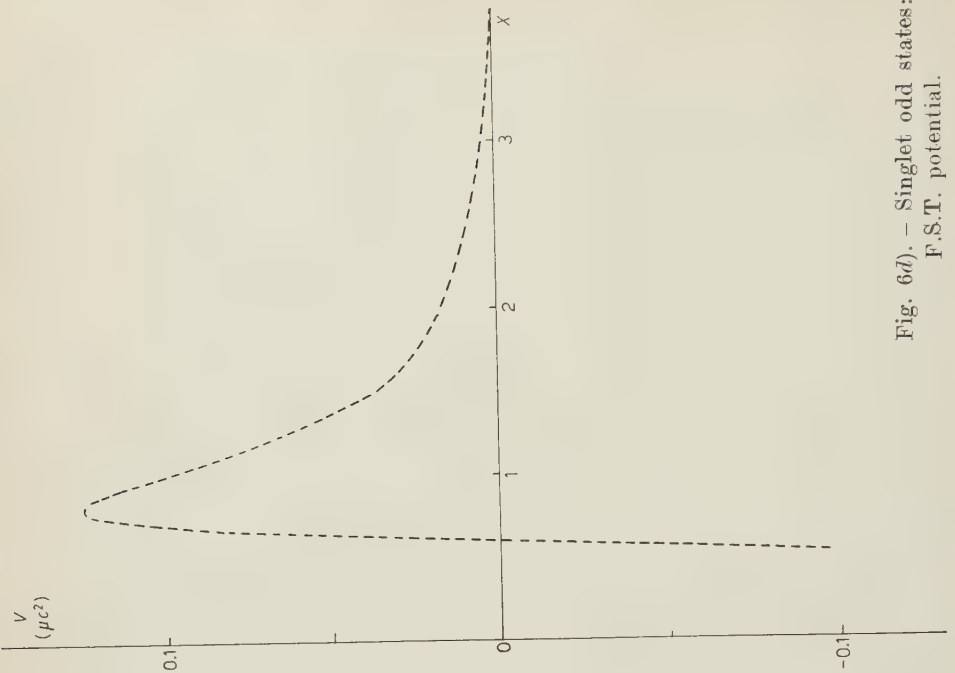Fig. 6a). – Triplet even states: F.S.T. potential (- - -central part. ... tensor part).

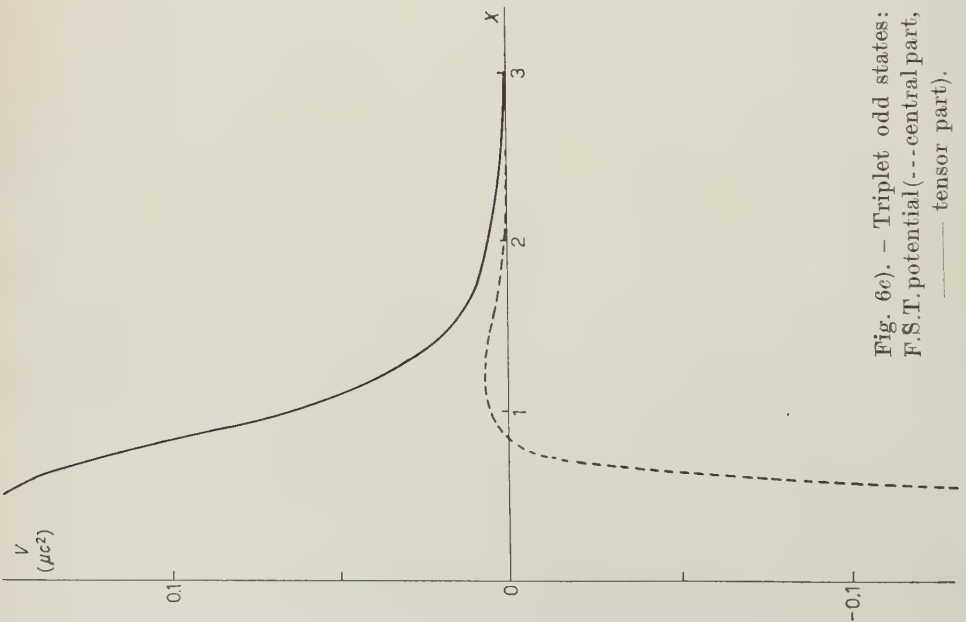Fig. 6d). – Singlet odd states:
F.S.T. potential.

Fig. 6e). – Triplet odd states:
F.S.T. potential (- - - central part,
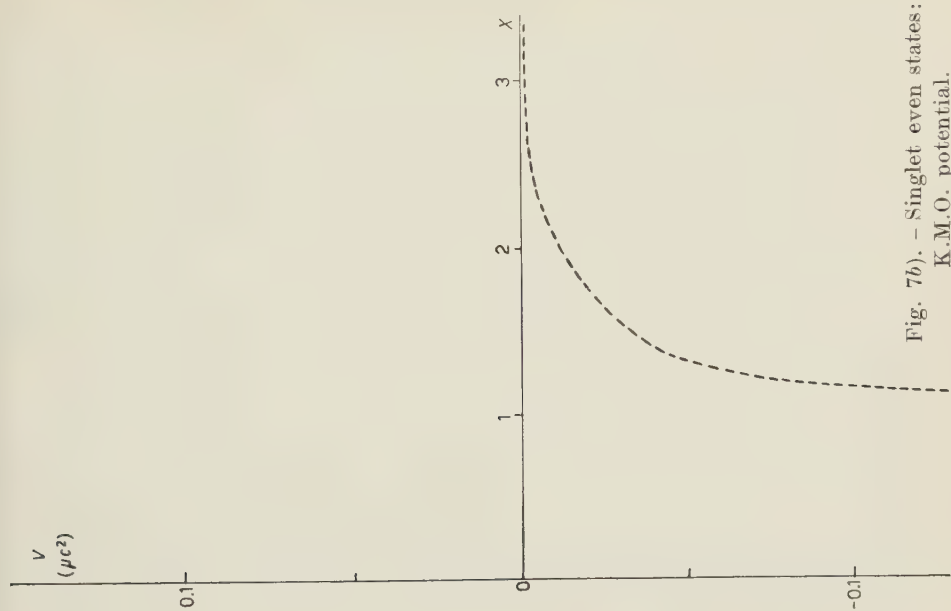————— tensor part).
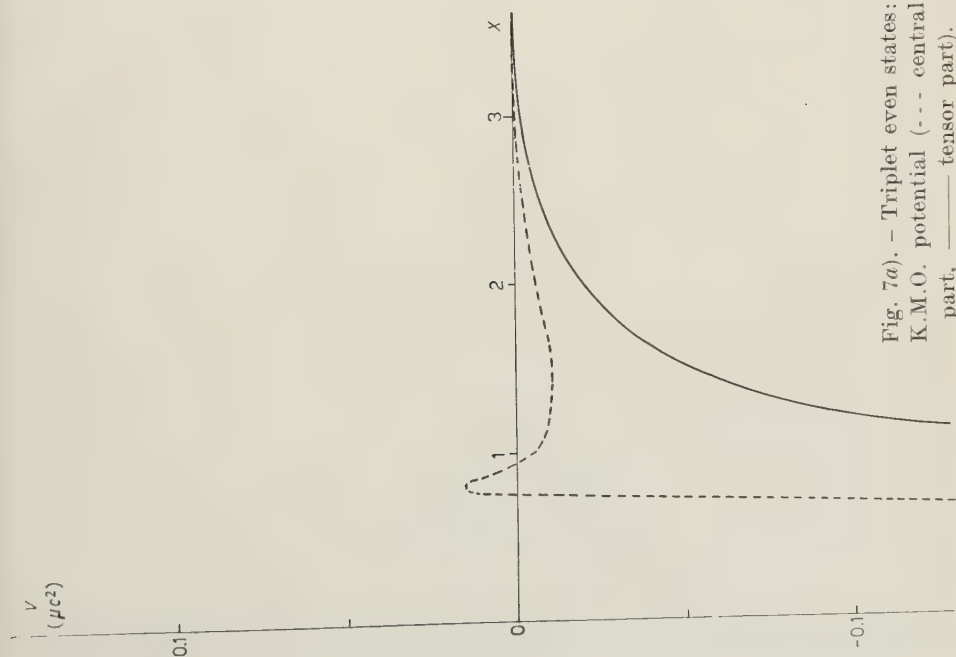
Fig. 7b). – Singlet even states: K.M.O. potential.

Fig. 7a). – Triplet even states: K.M.O. potential (- - - central part, ——— tensor part).
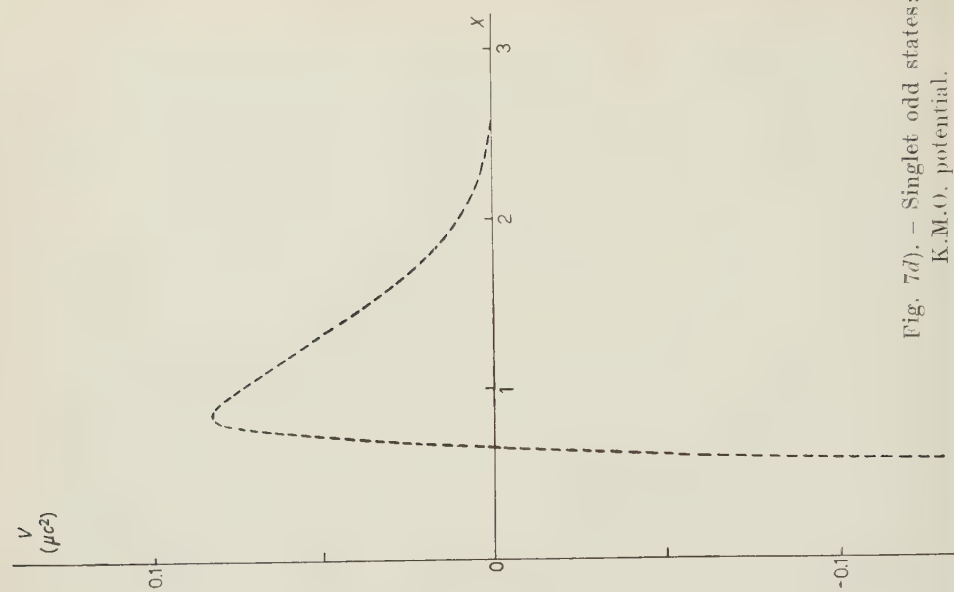
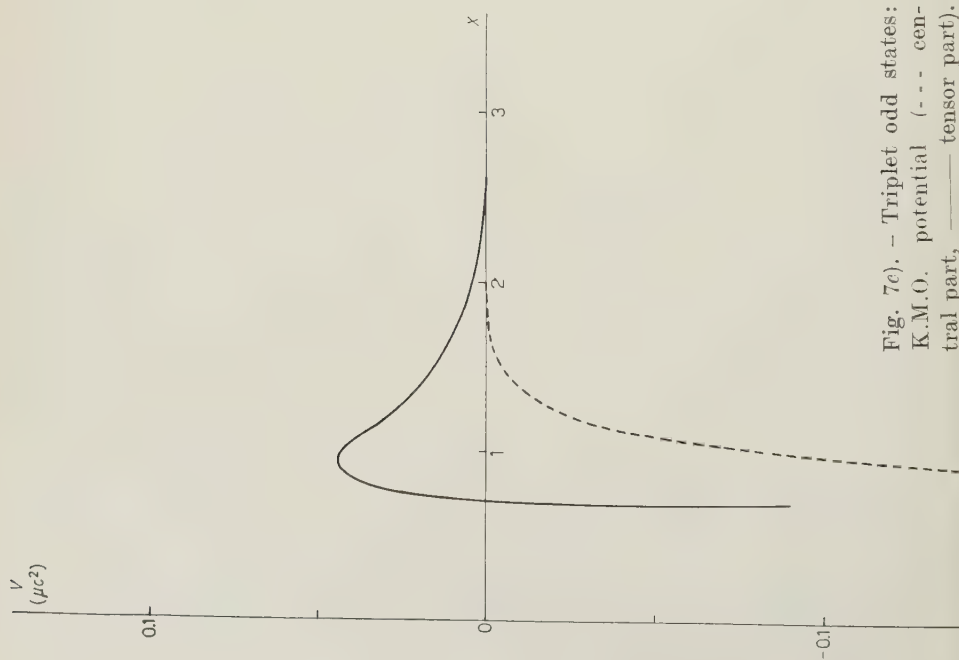Fig. 7d). – Singlet odd states:
K.M.O. potential.

Fig. 7c). – Triplet odd states:
K.M.O. potential (- - - cen-
tral part, ——— tensor part).

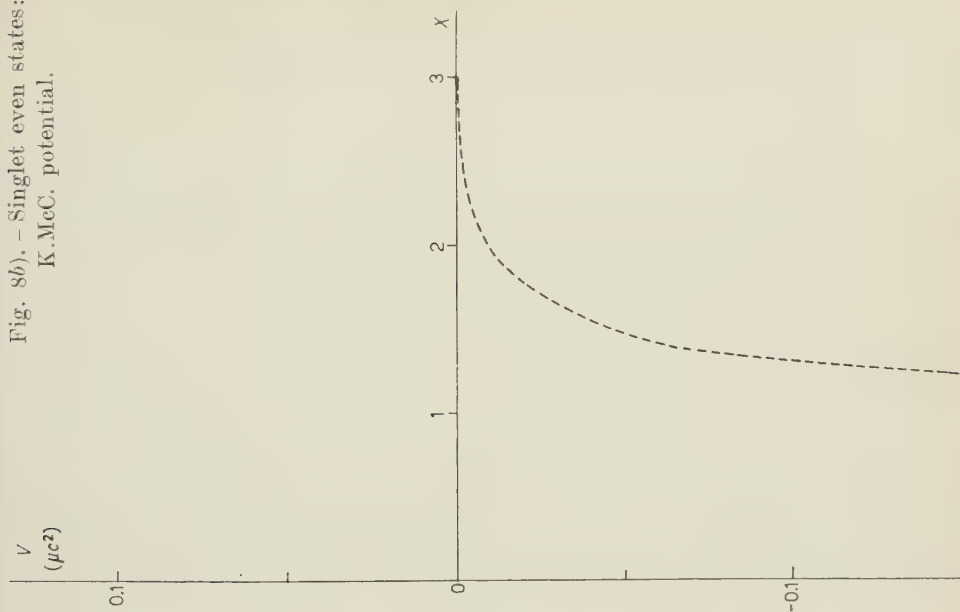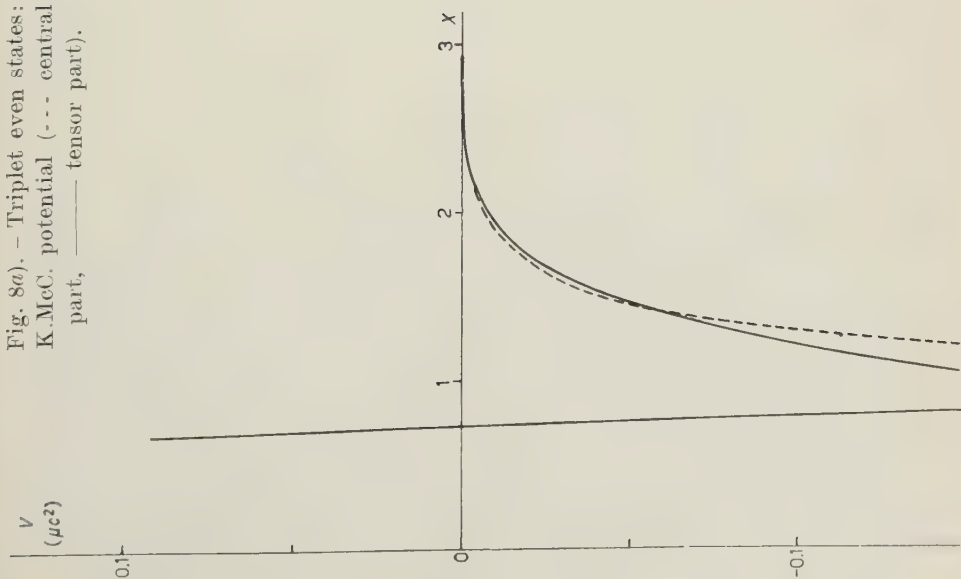Fig. 8b). – Singlet even states: K.McC. potential.

Fig. 8a). – Triplet even states: K.McC. potential (- - - central part, ——— tensor part).

two-pion exchange potential is ripe enough to make these tests very meaningful; we refer therefore to the original literature for these tests ([81]).

<div align="center">* * *</div>

([81]) For instance: S. FUJII, J. IWADARE, S. OTSUKI, M. TAKETANI, S. TANI and W. WATARI: *Progr. Theor. Phys.*, **11**, 11 (1954) for the T.M.O. potential; K. A. BRUECKNER and K. M. WATSON: *Phys. Rev.*, **92**, 1023 (1953); L. HULTHÉN and M. SUGAWARA: *Handbuch der Physik*, vol, XXXIX (Berlin, 1957) for the B.W. potential; S. GARTENHAUS: *Phys. Rev.*, **100**, 900 (1955); J. L. GAMMEL and R. M. THALER: *Phys. Rev.*, **103**, 1874 (1957) for the Gartenhaus potential; M. KONUMA, H. MIYAZAWA, and S. OTSUKI: *Progr. Theor. Phys.*, **19**, 17 (1958) for the K.M.O. potential.

Overall comparisons for the various potentials have been made by Japanese researchers and may be found in several articles of the Progress of Theoretical Physics.

An Outline of a Mathematical Theory of PAPA.

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CONTENTS. — **1**. Introduction. – **2**. Algebra of patterns and masks. – **3**. Errors in the machine. – **4**. Analogy with the mathematics of quantum mechanics. – APPENDIX.

1. – Introduction.

A learning machine, called PAPA (italian abbreviation for Automatic Programmer and Analyser of Probabilities) has been built in our Institute according to the suggestions of one of us (A.G.) [1]. Very briefly it works as follows.

A set of photocells (A-units) receive the image of the pattern to be shown as filtered by a random mask on top of each photocell. According to whether the total amount of light is greater or smaller than the amount of light falling on a reference cell with an attenuator, the photocell will fire a « yes » or « no » answer into the « brain » part of the PAPA. The latter is simply a memory storing the « yes » and « no » frequencies of excitation of each A-unit for each class of patterns shown, together with a computing part that « multiplies » (or « adds logarithms ») in order to evaluate the probability that an unknown pattern belongs to a given class. For details, the reader is referred to the papers quoted in footnote [1].

The purpose of the present note is to discuss some of the mathematics involved in the machine.

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[1] A. GAMBA: *Proc. IRE.*, **49**, 349 (1961); G. PALMIERI and R. SANNA: *Methodos*, to be published; A. GAMBA, L. GAMBERINI, G. PALMIERI and R. SANNA: *Suppl. Nuovo Cimento*, **20**, 112 (1961); A. GAMBA, G. PALMIERI and R. SANNA: *Suppl. Nuovo Cimento*, **20**, 146 (1961).

2. – Algebra of patterns and masks.

In discussing the PAPA machine we have to consider both patterns and masks.

Patterns are the objects shown to the machine. We may assume for simplicity that patterns are quantized like the following (20-quantized) example

(1)

where the black and white zones are those that in the actual pattern are, respectively, dark and bright. Masks are those that appear on top of each photocell as filter and can be represented by symbols of the same type as the patterns, with the only difference that the black and white zones will now be interpreted as opaque and transparent zones, respectively. We will consider only quantized masks, too. Continuous patterns and masks will be considered as limits of quantized ones. Patterns and masks will therefore be identified by the same kind of symbols.

Using the convention of reading the successive black and white zones like the lines of a page and using the correspondence

(2)
$$\begin{cases} \text{bright (transparent)} \to 1 \\ \text{dark (opaque)} \qquad \to 0 \end{cases}$$

we can use a shorthand notation. Thus in the case of example (1)

(3) (10110100011110100111) .

This convention will also allow us to introduce a Boolean algebra which holds for both patterns and masks.

N-quantized symbols like (3)—which may represent either a pattern or a mask—will be denoted in the following by small greek letters α, β, γ, ... The « threshold of a symbol » is defined as the number of 1's appearing in the symbol independently of the order and will be indicated as an index to the greek letter used for the symbol; thus the symbol in (3) will be indicated as α_{12} (or β_{12}, γ_{12}, ...). We may further define the product of two (N-quantized) symbols as the ordered Boolean product of the corresponding quantized « points ». Thus for $N = 6$, we have for example

(4) $(100110) \cdot (101101) = (100100)$

or shortly

(5) $\alpha_3 \beta_4 = \gamma_2$.

The relevance of this definition of product is apparent: when in PAPA a pattern (α_3) is shown through a mask (β_4) to an A-unit, the total amount of light falling into the photocell is equal to the threshold of the product (γ_2). Hence the usefulness of defining such a product and the reason for calling threshold the quantity which is the relevant one to decide whether the physical threshold of the A-unit will be exceeded or not.

The following manipulation rules are easily proved:

$$(6) \qquad \alpha_r \alpha_r = \alpha_r^2 = \alpha_r$$

i.e. the symbols are «idempotent» or, as a physicist would prefer to say, «projection operators»,

$$(7) \qquad (\alpha_r \beta_s)\gamma_t = \alpha_r(\beta_s\gamma_t) ,$$

$$(8) \qquad \alpha_r \beta_s = \beta_s \alpha_r ,$$

i.e. the product is both associative and commutative. For an arbitrary β_k:

$$(9) \qquad \alpha_N \beta_k = \beta_k ,$$

$$(10) \qquad \alpha_0 \beta_k = \alpha_0 ,$$

i.e. there exists the «unit» symbol

$$(11) \qquad \alpha_N = (111 \ldots 1) = \mathbf{1}$$

and the «zero» symbol

$$(12) \qquad \alpha_0 = (000 \ldots 0) = \mathbf{0} .$$

From the definition follows that the product of two N-quantized symbols with threshold r and s, respectively, is a symbol with a threshold t, such that

$$(13) \qquad t_2 = \min (r, s) \geqslant t \geqslant \max (r + s - N, 0) = t_1 .$$

A more relevant formula for products will now be obtained. Let S_r be the set of all the $\binom{N}{r} = \dfrac{N!}{(N-r)!\,r!}$ different symbols with threshold r. Let $S_r \cdot S_s$ be the product of two such sets to be understood in the sense that every symbol of the set S_r is to be multiplied by every symbol of the set S_s. Then the $\binom{N}{r} \cdot \binom{N}{s}$ products may be collected into sets S_t and the following theorem holds

$$(14) \qquad S_r \cdot S_s = \sum_{t=t_1}^{t_2} \frac{(N-t)!}{(r-t)!\,(s-t)!\,(N-r-s+t)!} S_t ,$$

where the sum is to be understood as «the entire set S_t appears as many times as indicated by its coefficient». The proof of formula (14) is given in the Appendix.

Taking into account the number $\binom{N}{t}$ of symbols of threshold t and the normalization coefficient

$$(15) \qquad \sum_{t=t_1}^{t_2} \frac{N!}{(r-t)!(s-t)!(N-r-s+t)!t!} = \binom{N}{s}\binom{N}{r},$$

we may interpret formula (14) by saying that by *multiplying a symbol of threshold r by a symbol of threshold s, one has a probability*

$$(16) \qquad \frac{r!s!(N-r)!(N-s)!}{N!(r-t)!(s-t)!t!(N-r-s+t)!},$$

of getting a symbol of threshold t.

We already mentioned that in PAPA a pattern—that without loss of generality we may assume to have a threshold $r \leqslant N/2$—is shown to a photocell through a mask of threshold s. When the threshold t of the product is above a certain value t' this A-unit is excited. The state of excitation of a given A-unit constitutes the information which is carried into the machine. If nothing is known about the nature of the patterns to be shown to PAPA, *i.e.* for a general purpose machine, it is clear that maximum information—in the sense of Shannon's theory of communication—is obtained from an A-unit when the probability of its excitation is $\frac{1}{2}$ for a random pattern. Then the amount of information obtained is exactly one bit. This situation can most easily be obtained by putting the threshold s of the mask equal to $N/2$ and the triggering threshold t' of the A-unit equal to $r/2$. This follows from formula (16) which is symmetric in t around the point $t = r/2$ for $s = N/2$.

From now on the symbol corresponding to the mask of the k-th A-unit will be denoted as follows

$$(17) \qquad \alpha_{N/2}(k) \cdot \alpha_{N/2}(l) = \gamma_{N/4}(k,\ l),$$

so that the threshold $N/2$ is clearly shown.

The above condition refers to an individual A-unit. However the PAPA is a set of n such A-units, $A_1, A_2, ..., A_n$. To get maximum information not only from each individual A-unit, but also from the whole set of them, we must require that the different A-units be uncorrelated, *i.e.* that the probability of excitation of the k-th A-unit, A_k, be independent of whether any other arbitrary A-unit, say A_l, is excited or not [2]. Obviously this condition

---

[2] This independence is also necessary eventually to evaluate total probabilities as products of elementary (uncorrelated) probabilities.

is satisfied if the transparent part of the unit A_k covers exactly half of the transparent part of the unit A_l, *i.e.* if the product of the masks of the corresponding A-units gives a symbol $\gamma_{N/4}$ of threshold $N/4$:

$$(18) \qquad\qquad \alpha_{N/2}(k) \cdot \alpha_{N/2}(l) = \gamma_{N/4}(k, l) \ .$$

A set of n A-units, such that any pair of their masks satisfies eq. (18), will be called an « orthogonal » set of A-units. The fixing of the threshold at $N/2$ (expression (17)) will be referred to as « normalization » of the A-units. From the above discussion it is clear that an « orthonormal » set of n A-units is a « best » set of n A-units for *unspecified* patterns, *i.e.* for a general purpose PAPA, with fixed masks and thresholds. One may now ask what is the maximum number n of orthonormal A-units for A-quantized symbols. Since an N-quantized symbol contains N bits of information and since every orthogonal A-unit picks up one bit, it is clear that n cannot be greater than N [3]. Let assume we take two random N-quantized symbols of threshold $N/2$. What is the probability that they are orthogonal? Using expression (16) for $r = s = N/2$ and $t = N/4$, we can easily see—for example using Stirling's formula—that as N increases this probability tends to *one*. A random set is a set of masks almost orthogonal. This justifies our choice of random masks in PAPA,

---

[3] The exact number, writing $N = 2^a \cdot b$, where b is odd, is $n = 2^a - 1$ (A. Levi and G. Manuzio: private communication). b occurs with quantized points, since one cannot split an odd number of points exactly into two. Leaving aside this trivial « unphysical » case, what is significant is the fact that n is « practically » of the same order of N. A « complete » set of 15 orthonormal symbols for $N = 16$ is given below as an example:
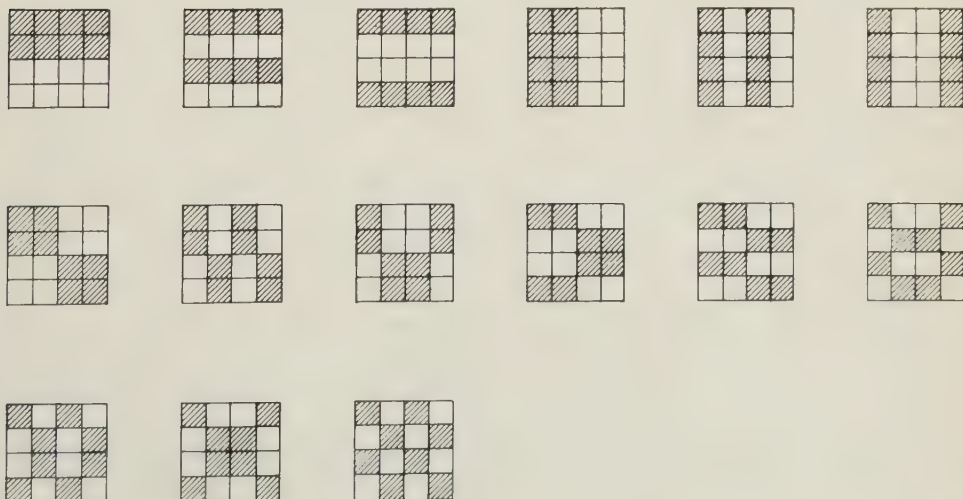
instead of an orthogonal set, apart from consideration of practical convenience one has to keep in mind if PAPA devices will ever be built on a (chemical?) microscopic scale.

3. – Errors in the machine.

For the sake of simplicity let us consider only discrimination between two classes a and b. Let us indicate with $A_1, A_2, A_3, ..., A_k, ..., A_n$ the associative units. After the learning process has been completed, PAPA will have memorized the following Table I, where a_k, b_k are the (relative) frequencies of excitation of the unit A_k, for the examples shown to the machine during the learning process in the classes a and b respectively $(0 < a_k < 1; \ 0 < b_k < 1)$ The corresponding frequencies of unexcitation will be denoted with bars; thus

(19) $$\bar{a}_k = 1 - a_k ; \qquad \bar{b}_k = 1 - b_k .$$

TABLE I.

A-unit Classes	A_1	A_2	A_3	...	A_k	...	A_n
a	a_1	a_2	a_3	...	a_k	...	a_n
b	b_1	b_2	b_3	...	b_k	...	b_n

An unknown symbol X, belonging either to class a or to class b, and with an excitation function E_x like the following example

(20)

A-units	A_1	A_2	A_3	A_4	...	A_n
E_x	1	0	1	1	...	0

will be recognized as belonging to class a when

(21) $$a_1\bar{a}_2a_3a_4 ... \bar{a}_n > b_1\bar{b}_2b_3b_4 ... \bar{b}_n$$

and to class b when the other inequality holds:

(22) $$a_1\bar{a}_2a_3a_4 ... \bar{a}_n < b_1\bar{b}_2b_3b_4 ... \bar{b}_n .$$

What is the probability that PAPA will make a mistake, when it has learned in the above sense?

Introduce the following abbreviation

(23)
$$\begin{cases} a_E = a_1\bar{a}_2 a_3 a_4 \dots \bar{a}_n \,, \\ b_E = b_1\bar{b}_2 b_3 b_4 \dots \bar{b}_n \,, \end{cases}$$

the index E indicating a particular number and ordering of quantities with and without bars, *i.e.* a particular excitation function; thus to consider all possible cases E could be an index running from 1 to 2^n.

a_E is the probability of occurrence of the excitation function E for patterns of class a; similarly for b_E.

$(a_E + b_E).2$ is the probability of occurrence of the excitation function E for patterns of class a and b when the occurrence of each class is equally likely. Let

(24)
$$\begin{cases} M_E = \max (a_E, b_E) \,, \\ m_E = \min (a_E, b_E) \,, \end{cases}$$

Then from the inequalities (21) and (22) it follows that $M_E/2$ is the probability of a correct identification—and $m_E/2$ the probability of a wrong identification—of a pattern with excitation function E when both classes a and b are *a priori* equally likely. The overall probability of a correct identification is then

(25)
$$R = \sum_E \frac{M_E}{2} \,,$$

and the overall probability of a « mistake » is

(26)
$$\varepsilon = \sum_E \frac{m_E}{2} \,,$$

where of course

(27)
$$R + \varepsilon = 1 \,.$$

It would be interesting to have an explicit expression for (25) in the form

(28)
$$R = R(n, \alpha, \beta) \,,$$

where n is the number of A-units and α and β are the standard deviation of the frequencies a_k and b_k respectively around the point $\frac{1}{2}$ (assuming a normal distribution). Then by measuring experimentally α and β one could derive the number n of A-units necessary for a PAPA with a preassigned reliability R. This mathematical problem is dealt with by E. GAGLIARDO in another paper.

4. – Analogy with the mathematics of quantum mechanics.

There is an analogy between the process of computing probabilities used in PAPA and the method of « second quantization » in physics.

Let the excitation function E_x of pattern X in example (20) be written in the form

$$(29) \qquad |E_x\rangle = |1, 0, 1, 1, ..., 0\rangle ,$$

$|E_x\rangle$ can be considered as a « ket » of a system of fermions (the « occupation numbers » being only 0's and 1's) with n independent states. Introducing the « creation operators » $A_1^\dagger, A_2^\dagger, ..., A_n^\dagger$, we can write

$$(30) \qquad |E_x\rangle = A_1^\dagger A_3^\dagger A_4^\dagger ... |0\rangle ,$$

where

$$(31) \qquad |0\rangle = |0, 0, 0, ..., 0\rangle$$

is the « vacuum state ». Similarly any other excitation function can be expressed by means of a ket of the form (29) or (30).

We can then consider the whole set of patterns of class a or class b (see Table I) as the kets

$$(32) \qquad \begin{cases} |a\rangle = \prod_k [a_k A_k^\dagger + (1 - a_k)]|0\rangle \\ \text{and} \\ |b\rangle = \prod_k [b_k A_k^\dagger + (1 - b_k)]|0\rangle , \end{cases}$$

respectively. Expressions (23), giving the probabilities that a pattern X with excitation function E_x belongs to class a and b are then given by the quantum mechanical « transition amplitudes »

$$(33) \qquad \langle E_x | a\rangle \qquad \text{and} \qquad \langle E_x | b\rangle ,$$

respectively. The class kets (32) are obtained from the kets corresponding to the excitation function of each individual pattern of the class, by a process different from the quantum mechanical process of the « superposition principle », due to the lack of coherence of our kind of superposition. A better

formal analogy is obtained if the correspondence is established with « density matrices », which can take into account also uncoherent superposition. The analogy then runs as follows.

Introduce for each A-unit, say A_k, the operator

$$(34) \qquad\qquad A_k = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

the eigenvalue 1 corresponding to the excited state and the eigenvalue 0 to the unexcited state of the unit. One can then define a one-to-one correspondence between excitation function E_r and density matrix ϱ in the product spaces of the operators (34). Thus in the case of the example (20) the excitation function would be represented as follows

$$(35) \qquad \varrho_x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdots \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

the density matrices

$$(36) \quad \begin{cases} \varrho_a = \begin{pmatrix} a_1 & 0 \\ 0 & 1-a_1 \end{pmatrix} \begin{pmatrix} a_2 & 0 \\ 0 & 1-a_2 \end{pmatrix} \cdot \begin{pmatrix} a_3 & 0 \\ 0 & 1-a_3 \end{pmatrix} \cdots \begin{pmatrix} a_n & 0 \\ 0 & 1-a_n \end{pmatrix} \\ \text{and} \\ \varrho_b = \begin{pmatrix} b_1 & 0 \\ 0 & 1-b_1 \end{pmatrix} \cdot \begin{pmatrix} b_2 & 0 \\ 0 & 1-b_2 \end{pmatrix} \cdot \begin{pmatrix} b_3 & 0 \\ 0 & 1-b_3 \end{pmatrix} \cdots \begin{pmatrix} b_n & 0 \\ 0 & 1-b_n \end{pmatrix} \end{cases}$$

would then represent the density matrices for the classes a and b respectively. In this case, however, they are obtained as superposition of the density matrices corresponding to the individual patterns belonging to the class, where superposition is now defined exactly like in quantum mechanics. The probabilities (33) are now given by the traces

$$(37) \qquad\qquad \mathrm{Tr}\,(\varrho_x \varrho_a) \qquad \text{and} \qquad \mathrm{Tr}\,(\varrho_x \varrho_b),$$

respectively.

We wanted to point out this analogy with quantum mechanics, not only for the sake of mathematics, but also for its physical interest. These considerations in fact suggest that quantum mechanical PAPA devices of submicroscopic size may eventually be discovered.

APPENDIX

Proof of formula (14) of the text.

Consider the set S_{N-1} of all the (N-quantized) symbols of threshold $N-1$. They are N in number and they form a « generating set » for the totality of the symbols, since every symbol of threshold r can be obtained as a product of $N-r$ such symbols.

The following special case of formula (14) is easily proved by inspection

$$(1') \qquad S_{N-1} \cdot S_r = (N-r)S_r + (N-r-1)S_{r-1} ,$$

where the convention $S_{-|k|} = 0$ is introduced here and in the following to deal also with the special case $r = 0$.

In fact there are $N-r$ symbols of the generating set having their « zero » in those $N-r$ points where every symbol of threshold r has a zero. These products leave the symbol unchanged and this accounts for the first term in the right-hand side of eq. (1'). The other r symbols of the generating set will put a « zero » in place of a « one » and so transform the symbols from threshold r to threshold $r-1$. The totality of symbols of threshold $r-1$ (and therefore the set S_{r-1}) must be obtained by symmetry: the number of times this set appears is then determined by the requirement that the number of products on both sides of eq. (1') be the same.

Multiplying formula (14) on both sides by S_{N-1} and using eq. (1'), one obtains after collecting terms

$$(2') \qquad S_{r-1} \cdot S_s = \sum_{t=t_1}^{t_2} \left\{ \frac{(N-t)!(r-t)}{(r-t)!(s-t)!(N-r-s+t)!(N-r+1)} S_t + \right.$$

$$\left. + \frac{(N-t+1)!}{(r-t)!(s-t)!(N-r-s+t)!(N-r+1)} S_{t-1} \right\} =$$

$$= \sum_{t=t_1}^{t_2} \frac{(N-t)!}{(r-t-1)!(s-t)!(N-r-s+t+1)!} S_t +$$

$$+ \frac{(N-t_2)!(r-t_2)}{(r-t_2)!(s-t_2)!(N-r-s+t_2)!(N-r+1)} S_{t_2} +$$

$$+ \frac{(N-t_1+1)!}{(r-t_1)!(s-t_1)!(N-r-s+t_1)!(N-r+1)} S_{t_1-1} .$$

The last term in eq. (2') is equal to

$$\begin{cases} 0 & \text{if } t_1 = 0 , \\[2mm] \dfrac{(2N-r-s+1)!}{(N-s)!(N-r+1)!} S_{r+s-N-1} & \text{if } t_1 = r+s-N , \end{cases}$$

the latter expression being nothing but the value under the sum in eq. (2′) for $t = r + s - N - 1$. We can therefore suppress this term and compensate for it by shifting the lower limit in the sum from t_1 to $t_1' = \max(r - 1 + s - N, 0)$.

Similarly the term in S_{t_2} is equal to

$$
\begin{cases}
0 & \text{if } t_2 = r\,, \\[2ex]
\dfrac{(N - s)!}{(r - s - 1)!(N - r + 1)!}\, S_s & \text{if } t_2 = s\,.
\end{cases}
$$

This term can also be put into the sum by shifting the upper limit from t_2 to $t_2' = \max(r - 1, s)$.

Therefore we can write

$$(3')\qquad S_{r-1}\cdot S_s = \sum_{t=t_1'}^{t_2'} \frac{(N - t)!}{(r - 1 - t)!(s - t)!(N - r + 1 - s - t)!}\, S_t\,,$$

which is nothing but eq. (14) with $r - 1$ in place of r.

Since the formula is true for $r = N - 1$ and s arbitrary (see eq. (1′)) the theorem is now proved by induction.

On the Evaluation of a Formula for the Errors of a Learning Machine.

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(ricevuto il 7 Luglio 1961)

1. – In a recent paper by A. Borsellino and A. Gamba [1] the computation of error in a learning machine gives rise to a statistical problem which can be stated in the following way: Let n be a given integer number (very large) and let

$$a_1, \dots, a_n, b_1, \dots, b_n,$$

be $2n$ real numbers with $0 < a_i < 1$, $0 < b_i < 1$.

Let

$$y = A \exp\left[-(x-\tfrac{1}{2})^2/\alpha^2\right], \qquad y = B \exp\left[-(x-\tfrac{1}{2})^2/\beta^2\right]$$

be the statistical distributions of the numbers a_i, b_i in the interval $(0, 1)$ where α, β are given constants, while A, B are the factors of normalization:

$$A = \left(\int_0^1 \exp\left[-(x-\tfrac{1}{2})^2/\alpha^2\right] \mathrm{d}x\right)^{-1}, \qquad B = \left(\int_0^1 \exp\left[-(x-\tfrac{1}{2})^2/\beta^2\right] \mathrm{d}x\right)^{-1}.$$

We denote by

$$\{\Phi_s\} \equiv \{\varphi_{s,1}(x), \dots, \varphi_{s,n}(x)\}, \qquad (s = 1, 2, 3, \dots, 2^n)$$

the totality of the sequences of n functions each of which can assume one or

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[1] A. Borsellino and A. Gamba: *Suppl. Nuovo Cimento*, **20**, 221 (1961).

the other of the two following determinations:

$$\varphi_{s,i}(x) = \begin{cases} x, \\ 1 - x. \end{cases}$$

For every s we consider the numbers:

$$M_s = \max \left[\varphi_{s,1}(a_1) \cdots \cdot \varphi_{s,n}(a_n), \; \varphi_{s,1}(b_1) \cdots \cdot \varphi_{s,n}(b_n)\right],$$

$$m_s = \min \left[\varphi_{s,1}(a_1) \cdots \cdot \varphi_{s,n}(a_n), \; \varphi_{s,1}(b_1) \cdots \cdot \varphi_{s,n}(b_n)\right].$$

Our problem is to calculate for given values of a, b, n, the two following numbers:

$$R = \tfrac{1}{2} \sum_{s}^{1...2^n} M_s, \qquad \varepsilon = \tfrac{1}{2} \sum_{s}^{1...2^n} m_s.$$

It can be noted that $R + \varepsilon = 1$, in fact for each sequence of n numbers $c_1, ..., c_n$ one has

$$\sum_{s}^{1...2^n} \varphi_{s,1}(c_1) \cdots \cdot \varphi_{s,n}(c_n) = 1,$$

as can be easily proven by induction with respect to n.

It can be also proven (see Section **4**) that for $n \to +\infty$ one has $\varepsilon \to 0$. The most important aspect of our problem is to determine the size of ε for given n, α, β.

Consider, for every fixed $0 \leqslant v \leqslant n$, the following 2^n vectors of the (x, y) plane:

$$v_s^{(v)} \equiv \left(1 \cdot \varphi_{s,1}(a_1) \cdots \cdot \varphi_{s,v}(a_v), \; 1 \cdot \varphi_{s,1}(b_1) \cdots \cdot \varphi_{s,v}(b_v)\right).$$

When speaking of vectors, it is always understood that they are applied at the origin.

For $v = 0$ all these vectors coincide with the vector $(1, 1)$. However, for $v = 1$ one has two positions generally distinct one from the other: (a_1, b_1), $(1 - a_1, 1 - b_1)$, which together, by addiction, give the vector $(1, 1)$; and as v increases every position breaks down into two others. The problem now is to study the distribution of the vectors $v_s^{(v)}$, in fact the value of 2ε is obviously given by the sum of the abscisses of the vectors $v_s^{(n)}$ which are found above the diagonal $y = x$ and of the ordinates of those vectors below. (By analogy we can calculate $2R$ by inverting the roles.) We can also take into consideration the symmetry relative to the diagonal.

For $v = 1$, the statistical distribution of the extreme points of vectors $v_s^{(1)}$

is given by the function

$$f(x, y) = \begin{cases} AB \exp\left[-\left(x-\tfrac{1}{2}\right)^2/\alpha^2 - \left(y-\tfrac{1}{2}\right)^2/\beta^2\right] & \text{for } 0 < x < 1, \ 0 < y < 1, \\ 0 & \text{elsewhere .} \end{cases}$$

The statistical distribution of the vectors $v_s^{(\nu+1)}$ which derive from the decomposition of a vector $v_s^{(\nu)} = (\xi, \eta)$ is given by

$$\frac{1}{\xi\eta} f\left(\frac{x}{\xi}, \frac{y}{\eta}\right).$$

Our question can now be reduced to a problem containing a single variable by means of the following observation: The evaluation of ε (and of R) remains obviously unchanged if during the decomposition, for ω vectors with the extreme points coinciding at the point (x, y), (ω real positive), we substitute ωx vectors with the extreme points coinciding in the point projection of (x, y), from the origin, on the line $x = 1$, that is in the point $(1, y/x)$.

By systematically using this observation, it is enough to consider only vectors with extreme points on the half line $x = 1$, $y > 0$. For $\nu = 0$ there is still coincidence with the vector $(1, 1)$.

Going from ν to $\nu + 1$ the statistical distribution of the vectors $(1, y_0)$ coming from ω vectors $(1, \eta)$ can be obtained by integrating along the line $y = y_0 x$, with $d\varrho = \sqrt{1 + y_0^2}\, dx$, the function

$$\frac{1}{\eta} f\left(x, \frac{y}{\eta}\right),$$

multiplied by ωx (because of the preceeding observation), by the jacobian $\varrho = x\sqrt{1 + y_0^2}$, and by

$$\frac{d\vartheta}{dy_0} = \frac{1}{1 + y_0^2}$$

(thus justifying a successive integration with respect to y_0 instead of ϑ), and is then given by

$$\omega AB \frac{1}{\eta} \int_0^{h(\eta, y_0)} x^2 \exp\left[-\left(x-\frac{1}{2}\right)^2 \Big/ \alpha^2 - \left(\frac{y_0 x}{\eta} - \frac{1}{2}\right)^2 \Big/ \beta^2\right] dx,$$

where

$$h(\eta, y_0) = \min\left[1, \frac{\eta}{y_0}\right].$$

Consider the sequence of functions $F_\nu(y)$ constructed in the following way:

$$
\begin{cases}
(F_0(y) = \delta(y - 1), \qquad (\text{Dirac})) , \\[2mm]
F_1(y) = AB \int_0^{h(\eta, y)} x^2 \exp\left[-\left(x - \frac{1}{2}\right)^2 \Big/ \alpha^2 - \left(yx - \frac{1}{2}\right)^2 \Big/ \beta^2\right] \mathrm{d}x , \\[2mm]
\cdots \qquad \cdots \qquad \cdots \qquad \cdots \qquad \cdots \qquad \cdots \qquad \cdots \\[2mm]
F_{\nu+1}(y) = AB \int_0^{+\infty} F_\nu(\eta) \frac{\mathrm{d}\eta}{\eta} \int_0^{h(\eta, y)} x^2 \exp\left[-\left(x - \frac{1}{2}\right)^2 \Big/ \alpha^2 - \left(\frac{yx}{\eta} - \frac{1}{2}\right)^2 \Big/ \beta^2\right] \mathrm{d}x .
\end{cases}
$$

The solution of the problem considered at the beginning is obviously given by

$$
\varepsilon = 2^n \int_0^{+\infty} F_n(y) \, \mathrm{d}y = 2^n \int_0^1 y F_n(y) \, \mathrm{d}y .
$$

2. – We come now to several considerations useful in the numerical treatment of our problem.

The points of the positive y-axis can be sketched by the following values:

$$
q^{-k} , \ \ldots, \ q^{-1} , \quad 1 , \quad q , \ \ldots, \ q^k ,
$$

where q is sufficiently near to 1 and k is an integer sufficiently large. More precisely we shall identify with $y = q^i$ the points of the segment:

$$
\tfrac{1}{2}(q^{i-1} + q^i) < y < \tfrac{1}{2}(q^i + q^{i+1}) , \qquad\qquad (i = -k, \ldots, +k).
$$

For every $0 \leqslant \nu \leqslant n$ the function $2F^\nu(y)$ can then be sketched by $2k+1$ numbers:

$$
\vartheta_{-k}^{(\nu)} , \ \ldots, \ \vartheta_0^\nu , \ \ldots, \ \vartheta_{+k}^{(\nu)} ,
$$

that for $\nu = 0$ have the initial values

$$
\vartheta_{-k}^{(0)} = 0 , \ \ldots, \ \vartheta_{-1}^{(0)} = 0 , \ \ \vartheta_0^{(0)} = 1 , \ \ \vartheta_1^{(0)} = 0 , \ \ldots, \ \vartheta_k^{(0)} = 0
$$

and for $\nu > 0$ can be constructed according to the following rule:

$$
\vartheta_i^{(\nu+1)} = \sum_j^{h \ldots h} I_{i-j} \vartheta_j^{(\nu)} ,
$$

where

$$I_i = 2AB \frac{1}{2}\left(q - \frac{1}{q}\right) q^i \int_0^{\min[1,\, q^{-i}]} x^2 \exp\left[- (x - \tfrac{1}{2})^2/\alpha^2 - (q^i x - \tfrac{1}{2})^2/\beta^2\right] dx\,.$$

The limits of integration, if a, b are sufficiently small, can be substituted by $-\infty$, $+\infty$.

The solution of our problem is given by

$$\varepsilon = \sum_{i>0} \vartheta_i^{(n)} + \frac{\vartheta_0^{(n)}}{2} = \sum_{i<0} q^i\, \vartheta_i^{(n)} + \frac{\vartheta_0^{(n)}}{2}\,.$$

It is easily seen that the values $\vartheta_i^{(1)}$ (putting $\vartheta_i^{(1)} = 0$ for $i < -k$ and for $i > k$) furnish the coefficients I_i of the twice repeated transformation. Continuing in this way one arrives at the following rule:

$$\vartheta_i^{(2\nu)} = \sum_j^{-k \ldots k} \vartheta_{i-j}^{(\nu)}\vartheta_j^{(\nu)}$$

which is very convenient to use, after having calculated $\vartheta_i^{(1)}$, if n is very large.

3. – To diminish the error inevitably introduced by numerical calculations the following observations are useful in programming:

 a) At every step it is convenient to normalize the numbers $\vartheta_i^{(\nu)}$, dividing by $\frac{1}{2}\sum_i \vartheta_i^{(\nu)}(1+q^i)$.

 b) The value of q can be changed (more than once) in the course of the procedure: initially very near to 1, then when $\vartheta_{-k}^{(\nu)}$, $\vartheta_{+k}^{(\nu)}$ begin to become significant with respect to $\vartheta_0^{(\nu)}$, it is useful to substitute q^2 for q; then the values $\vartheta_{2i}^{(\nu)}$ must be placed in the boxes reserved for $\vartheta_i^{(\nu)}$, while the $\vartheta_i^{(\nu)}$ with i odd can be forgotten or, even better can be divided into two equal parts and placed in the two adiacent boxes before making the aforementioned displacement.

 c) Because of small numerical errors the quantities

$$\sum_{i>0} \vartheta_i^{(\nu)} + \frac{\vartheta_0^{(\nu)}}{2}\,, \qquad \sum_{i<0} q^i\, \vartheta_i^{(\nu)} + \frac{\vartheta_0^{(\nu)}}{2}\,.$$

which theoretically should remain equal, progressively differ one from another, slowly at the beginning and then, by consequence, more quickly.

Fortunately the rule for the construction of $\vartheta_i^{(2\nu)}$ is invariant by translation with respect to i: when displacing the $\vartheta_i^{(\nu)}$ in the boxes reserved for the $\vartheta_{i+h}^{(\nu)}$, the $\vartheta_i^{(2\nu)}$ also undergo a similar displacement.

It is possible to profit from this fact by making at each step a displacement in such a way to make the two quantities under consideration as equal as possible.

4. – The method explained in the Section 2 proves also that (as remarked in Section 1) for $n \to +\infty$ one has $\varepsilon \to \infty$. (The treatment by means of integrals, as in Section 1, is left to the reader.)

We can put $k = +\infty$ and q arbitrarily near 1.

The numbers $\vartheta_i^{(\nu)}$ satisfy the condition:

$$\tfrac{1}{2} \sum_i \vartheta_i^{(\nu)}(1 + q^i) = 1 \ ,$$

thus

$$\sum_i \vartheta_i^{(\nu)} \leqslant 2 \ .$$

Let

$$\eta_i^{(\nu)} = \vartheta_i^{(\nu)} \Big/ \sum_j \vartheta_i^{(\nu)},$$

it is enough to prove that for $\nu \to +\infty$

$$\sum_{i<0} q^i \eta_i^{(\nu)} + \frac{\eta_0^{(\nu)}}{2} \to 0 \ .$$

The rule which gives the $\eta_i^{(\nu)}$ is

$$\eta_i^{(\nu+1)} = \sum_j K_{i-j} \eta_j^{(\nu)} \ ,$$

where

$$K_i = \frac{I_i}{\sum_j I_j} \ .$$

Since there are at least two $K_i \neq 0$, it is easily seen that the center of the system of the weights $\eta_i^{(\nu)}$ at the points i of the x-axis, when progressing from ν to $\nu+1$, undergoes a constant translation (which could also be a vanishing one) while the center of the system of the same weights placed at the points $|i|$ diverges.

Hence the above statement.

5. – A brief calculation made by means of the IBM 650 at the Centro di Calcolo Numerico of the University of Genova has given the following results:

TABLE I.

n	$\varepsilon(n)$		
	$\alpha=\beta=1/10$	$\alpha=\beta=1/100$	$\alpha=\beta=1/1000$
$2 \div 2^2$	$4.44 \cdot 10^{-1}$	$4.94 \cdot 10^{-1}$	$4.994 \cdot 10^{-1}$
$2^2 \div 2^3$	$4.23 \cdot 10^{-1}$	$4.92 \cdot 10^{-1}$	$4.992 \cdot 10^{-1}$
$2^3 \div 2^4$	$3.94 \cdot 10^{-1}$	$4.88 \cdot 10^{-1}$	$4.988 \cdot 10^{-1}$
$2^4 \div 2^5$	$3.58 \cdot 10^{-1}$	$4.84 \cdot 10^{-1}$	$4.984 \cdot 10^{-1}$
$2^5 \div 2^6$	$3.11 \cdot 10^{-1}$	$4.77 \cdot 10^{-1}$	$4.97 \cdot 10^{-1}$
$2^6 \div 2^7$	$2.25 \cdot 10^{-1}$	$4.68 \cdot 10^{-1}$	$4.96 \cdot 10^{-1}$
$2^7 \div 2^8$	$1.94 \cdot 10^{-1}$	$4.56 \cdot 10^{-1}$	$4.95 \cdot 10^{-1}$
$2^8 \div 2^9$	$1.31 \cdot 10^{-1}$	$4.39 \cdot 10^{-1}$	$4.93 \cdot 10^{-1}$
$2^9 \div 2^{10}$	$7.57 \cdot 10^{-2}$	$4.16 \cdot 10^{-1}$	$4.91 \cdot 10^{-1}$
$2^{10} \div 2^{11}$	$3.46 \cdot 10^{-2}$	$3.86 \cdot 10^{-1}$	$4.87 \cdot 10^{-1}$
$2^{11} \div 2^{12}$	$1.14 \cdot 10^{-2}$	$3.47 \cdot 10^{-1}$	$4.82 \cdot 10^{-1}$
$2^{12} \div 2^{13}$	$2.40 \cdot 10^{-3}$	$2.99 \cdot 10^{-1}$	$4.75 \cdot 10^{-1}$
$2^{13} \div 2^{14}$	$2.63 \cdot 10^{-4}$	$2.41 \cdot 10^{-1}$	$4.65 \cdot 10^{-1}$
$2^{14} \div 2^{15}$	$1.15 \cdot 10^{-5}$	$1.78 \cdot 10^{-1}$	$4.51 \cdot 10^{-1}$
$2^{15} \div 2^{16}$	$1.39 \cdot 10^{-7}$	$1.16 \cdot 10^{-1}$	$4.32 \cdot 10^{-1}$
$2^{16} \div 2^{17}$	$2.68 \cdot 10^{-10}$	$6.39 \cdot 10^{-2}$	$4.07 \cdot 10^{-1}$
$2^{17} \div 2^{18}$	$3.87 \cdot 10^{-14}$	$2.72 \cdot 10^{-2}$	$3.74 \cdot 10^{-1}$
$2^{18} \div 2^{19}$	$1.43 \cdot 10^{-19}$	$8.18 \cdot 10^{-3}$	$3.31 \cdot 10^{-1}$
$2^{19} \div 2^{20}$	$3.00 \cdot 10^{-27}$	$1.48 \cdot 10^{-3}$	$2.80 \cdot 10^{-1}$
$2^{20} \div 2^{21}$	$4.14 \cdot 10^{-38}$	$1.33 \cdot 10^{-4}$	$2.20 \cdot 10^{-1}$
$2^{21} \div 2^{22}$	—	$4.43 \cdot 10^{-6}$	$1.56 \cdot 10^{-1}$
$2^{22} \div 2^{23}$	—	$3.58 \cdot 10^{-8}$	$9.70 \cdot 10^{-2}$
$2^{23} \div 2^{24}$	—	$3.94 \cdot 10^{-11}$	$4.92 \cdot 10^{-2}$
$2^{24} \div 2^{25}$	—	$2.57 \cdot 10^{-15}$	$1.88 \cdot 10^{-2}$
$2^{25} \div 2^{26}$	—	$3.10 \cdot 10^{-22}$	$4.84 \cdot 10^{-3}$

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